Analysis of Three-Body Problem Orbits with Application to the Lunar Deep Space Gateway

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This work explores the insertion problem in near rectilinear Halo orbits from Earth using impulsive maneuvers and then applies the results to the *Deep Space Gateway* (DSG) mission. The paper starts with a brief review of the three-body problem, from its more general conceptualization to the simplified version of the Circular Restricted Three Body Problem (CR3BP), which will be used to compute the Halo orbits with a numerical continuation tool, the AUTO software. The methodology for the computation of the insertion trajectories and the complete exploration of the optimization problem are described. Next, the results of applying this methodology to the DSG mission are shown, and finally some conclusions are derived.

Nomenclature

NRHO	=	Near Rectilinear Halo Orbit
DSG	=	Deep Space Gateway
CR3BP	=	Circular Restricted 3 Body Problem
R3BP	=	Restricted 3 Body Problem
BVP	=	Boundary Value Problem
LEO	=	Low Earth Orbits

I. Introduction

Leaded by NASA and supported by the main space agencies in the world (ESA, JAXA and CSA), the next step to advance in the exploration of space starts by building a new space station in cislunar space, the *Deep Space Gateway*. The principal goal of this station is for the astronauts and the agencies to test the systems needed for missions in deep space (such as Mars and the asteroids), and this area of the Earth-Moon system offers a great opportunity to gain experience. It could also serve as a middle point between Earth and a Moon base. For this space station to be of practical use it needs to be placed in an orbit such that it is easily accessible and relatively cheap to maintain, and at the same time it has to serve as the gateway for the future missions of exploration in the Solar System. The study of periodic motion in three-body problems [1] applied to the Earth-Moon system will serve as a first step to address the location of this mission.

The *Three Body Problem* (3BP) is simplified for the purpose of this work; first to the restricted version, R3BP, then to the *Elliptical Restricted* version, ER3BP, and finally to the *Circular* one, CR3BP, which is studied with AUTO to compute the Halo orbits, a family of periodic orbits known for having a peculiar shape as seen from Earth. There is also a great variety of Halo orbits themselves, and they have been extensively studied in the context of space exploration [2]. Inside this family of orbits there is a subset that stands out for the purpose of having an inhabited station with deep space access [3], which are the *Near Rectilinear Halo Orbits*, or NRHOs. These orbits are at the edge of Earth's gravity well from an energetic point of view, which makes them ideal as stepping stone for deep space missions, and their stability properties are quite promising.

The calculation of Halo orbits can be addressed analytically up to a certain point [4] [5] because of the non-linear nature of the problem, and this is usually the first step in the design of Halo orbits. Traditionally, many numerical approaches have been used to compute better approximations of these orbits in simplified models. [6] uses a variation of constants method to approximate Halo orbits in the *Elliptical Restricted* model that is shown on section 2. In [7] a differential correction method is used to refine analytical approximations in the CR3BP. The approach used in this work is that of numerical continuation [8], that allows to extend linear approximation of the periodic families near the equilibrium points to the complete system using the AUTO software as we see in section 3.

Targeting the desired Halo orbit is another problem that can be addressed in multiple ways. The study on the insertion in Halo orbits has been profoundly explored, and methods that include the use of manifolds to perform the insertion have already been considered [9], as well as lunar fly-bys to generate more general trajectories [10]. These works, on the other hand, focus on Halo orbits that are farther from the Moon than the NRHOs that we are considering for the DSG mission, and that makes some of the results not applicable, as the dynamics very close to the Moon (as it is the case for the most interesting NRHOs) changes dramatically the shape of the manifolds. [11] briefly contemplates the use of impulsive insertion for a NRHO of the L2 point, with a solution that, as will be shown, can be obtained with the method developed here. [12] also uses a direct insertion approach (2 impulses) for NRHOs in the H2 family.

In this work, many of the these possibilities are considered for NRHOs in both the H1 and H2 family, algorithms are developed with a global optimization mindset similar to heuristics for design of interplanetary trajectories [13], using the tools of the *Global Optimization Toolbox* that *Matlab* provides. Specifically, very interesting options that consider the use of *free-return trajectories* are proposed, with some of them being actually usable for a real manned mission scenario with enhanced safety measures.

The structure of this work is as follows. Section 2 is a review of the 3BP, as it is the core of the issue under study. Then, section 3 lays out the continuation methodology that has been used for the computation of the Halo orbits in this works, with a brief explanation of the internal workings of AUTO and the workflow with it. Section 4 explains the reasoning behind the methodology for the computation of the insertion trajectories and the optimization strategy. Finally, section 5 is a compilation of the most important results that the method has provided. The document ends with some conclusions and proposed future lines of work.

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II. The Three-Body Problem

The three-body problem is a classical problem in physics and mechanics that, in contrast to the Keplerian two-body problem, doesn't have an analytical solution in general. The gravitational problem dates from 1687, when Newton published his "Principia" and set out the problem of the movements of three masses subjected to the influence of each other, he also tried to apply his results to study the motion of the Moon under the influence of the Earth and the Sun. In 1747 Jean le Rond d'Alembert and Alexis Clairaut both submitted their analyses to the Académie Royale des Sciences, in which they used differential equations to be solved by successive approximations. The name "three-body problem" (Problème des Trois Corps) began to be used in the 1740s Paris due to this research. It was Lagrange in 1772 who demonstrated analytical solutions do exist if certain restrictions are imposed. Finally, in 1887, Heinrich Bruns and *Henri Poincaré* showed that no analytical solutions given by algebraic expressions and integrals can exist in general.

For the general case the system is 18th-order, and 10 analytical integrals can be written, corresponding to conservation of momentum (three integrals), energy (one integral), and motion of the system's mass center (six integrals). Together, this integrals and special case conditions allow for some solutions to be analytically studied (Lagrange's three-body Solutions [14]). Others special restricted forms of the problem have been studied, but so far, the previous cases mentioned are the only ones with analytical solution. Here, the sub-problem to be addressed is that of one of the masses to be negligible for the motion of the other two, so that it can be effectively divided in two separate problems: the problem of two masses exerting their gravitational influence on each other, and the problem of a third mass, the spacecraft, subjected to the influence of the other two. The first problem is well known and has an analytical solution, the two-body problem (2BP). This will allow for the second problem to be addressed and, as it restricts the value of the third mass to be very small, it is called the Restricted three body problem, or R3BP. This section will review the R3BP from its general conception to the even more simplified versions, developing in the path all the necessary equations for the studies that will be performed later.

Now, for the R3BP to be properly set up, we must first consider the motion of the 3 masses involved (M_1 and M_2, the primaries, and m the spacecraft) in an inertial frame, their positions given by $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}$ respectively. The equations of motion of any of them can be obtained in general through the direct the sum of the gravitational forces the others exert, as given by equations (1).

$$
\begin{aligned}
m\ddot{\boldsymbol{R}}\rfloor_I &= -G\frac{mM_1}{r_1^3}\boldsymbol{r}_1 - G\frac{mM_2}{r_2^3}\boldsymbol{r}_2,\\
M_1\ddot{\boldsymbol{R}}_1\rfloor_I &= G\frac{M_1M_2}{r_{12}^3}\boldsymbol{r}_{12} - G\frac{mM_1}{r_1^3}\boldsymbol{r}_1,\\
M_2\ddot{\boldsymbol{R}}_2\rfloor_I &= -G\frac{M_1M_2}{r_{12}^3}\boldsymbol{r}_{12} - G\frac{mM_2}{r_2^3}\boldsymbol{r}_2,
\end{aligned}
\tag{1}
$$

where $\boldsymbol{r}_1 = \mathbf{R} - \mathbf{R}_1$, $\mathbf{r}_2 = \mathbf{R} - \mathbf{R}_2$ and $\mathbf{r}_{12} = \mathbf{R}_2 - \mathbf{R}_1$. If we restrict the mass of the spacecraft to be negligible when compared to the primaries the motion of M_1 and M_2 cease to depend on the position of m, decoupling the first equation and leading to the R3BP. Supposing then that the primaries revolve around their common center of mass in elliptic orbits we are faced with the *Elliptic Restricted Three-Body Problem*, or ER3BP. The sub-solution to the 2BP of the primaries is analytically know, in particular their instantaneous angular velocity. This way, if we express the motion of m in a synodic frame of reference (S) that revolves with the primaries in its X axis, the equations of motion can be expressed as:

$$
\ddot{\boldsymbol{R}}\rfloor_S + 2\boldsymbol{w}_{S/I}\times\dot{\boldsymbol{R}}\rfloor_S + \dot{\boldsymbol{w}}_{S/I}\rfloor_S\times\boldsymbol{R} + \boldsymbol{w}_{S/I}\times(\boldsymbol{w}_{S/I}\times\boldsymbol{R}) = -\mu_1\frac{\boldsymbol{r}_1}{r_1^3} - \mu_2\frac{\boldsymbol{r}_2}{r_2^3}.
\tag{2}
$$

Denoting $R=[x\, y\, z]^T$ we can obtain the differential equations of the motion of m in S, see system (3).

$$
\begin{aligned}
&\ddot{x} - 2\dot{\theta}\dot{y} - \ddot{\theta}y - \dot{\theta}^2x = -\mu_1\frac{x+R_1}{r_1^3} - \mu_2\frac{x+R_2}{r_2^3},\\
&\ddot{y} + 2\dot{\theta}\dot{x} + \ddot{\theta}x - \dot{\theta}^2y = -\mu_1\frac{y}{r_1^3} - \mu_2\frac{y}{r_2^3},\\
&\ddot{z} = -\mu_1\frac{z}{r_1^3} - \mu_2\frac{z}{r_2^3}.
\end{aligned}
\tag{3}
$$

The equation that defines θ for all t is also needed, but in order to avoid this inconvenience a change in the independent variable will be done. We will do that while normalizing the equations system, so that it gets simpler and more suitable for numerical applications. To normalize distances we will use the instantaneous separation between the primaries, so that in the adimensionalized system it is always 1. For the independent variable we will use:

$$
dt = \frac{(1-e^2)^{3/2}}{n\Phi^2}d\theta\ ,
\tag{4}
$$

where

$$
\Phi = 1 + e\cos\theta,\quad n = \sqrt{\frac{G(M_1+M_2)}{a^3}} = \sqrt{\frac{\mu_T}{a^3}}
$$

with e the eccentricity and a the semi-mayor axis of the 2BP orbit. Applying this changes to the complete system results in:

$$
\begin{aligned}
\tilde{x}'' - 2\tilde{y}' &= \frac{1}{1+e\cos\theta}\left(\tilde{x} - (1-\mu)\frac{\tilde{x}+\mu}{\rho_1^3} - \mu\frac{\tilde{x}-(1-\mu)}{\rho_2^3}\right),\\
\tilde{y}'' + 2\tilde{x}' &= \frac{1}{1+e\cos\theta}\left(\tilde{y} - (1-\mu)\frac{\tilde{y}}{\rho_1^3} - \mu\frac{\tilde{y}}{\rho_2^3}\right),\\
\tilde{z}'' &= \frac{1}{1+e\cos\theta}\left(-\tilde{z}e\cos\theta - (1-\mu)\frac{\tilde{z}}{\rho_1^3} - \mu\frac{\tilde{z}}{\rho_2^3}\right).
\end{aligned}
\tag{5}
$$

Where ′ denotes derivative with respect to θ. A first glance at these equations reveals that the equilibria (Lagrange Points) in the problem normalized with $\mathbf{r}_{12}(\theta)$ is independent of e and θ, which means that the ratio of the distance between the stationary points (in the rotating frame) and the primaries is constant in the ER3BP. Linearizing the equations around the collinear points of equilibrium (along X axis) for the case of circular motion (e=0, CR3BP) shows that there are two different types of periodic motion that emerge from them, one following the in-plane oscillatory mode and the other the out-of-plane oscillatory mode. It is this fact that will allow for the computation of periodic orbits in the non-linearized CR3BP system, thanks to the continuation techniques that will be explained in the next section.

III. AUTO-based numerical continuation for the computation of Halo orbits

Henri Poincaré was one of the first mathematicians to notice that some problems reached such levels of complexity in its nature that the concept of solving the problem had to be changed just for those, as for a complete solution was well above what he believed achievable. Because of this difficulty, some problems must be faced in a different way, not looking for the solution in a form of an equation, all the information there at plain sight. The alternative showed here, continuation, presents itself as a very powerful and efficient tool. It relies in much of the work of Henri Poincaré (dynamical systems, topology, index theory, the use of the implicit function theorem, bifurcation theory...), as continuation can be used to obtain a global picture of the system under study, without solving its equations. Continuation will result in the structure of the solution as different key parameters change (bifurcation diagram), it will reveal dynamical structures hidden in its equations (equilibria, periodic orbits, invariant manifolds...) and the qualitative behavior around them.

The theoretical support for continuation can be found in [8], though the basis will be presented here plainly. This section will show the internal workings of the software AUTO in the continuation of solutions and periodic orbits, all of this directly applied to the problem at hand for the computation of Halo orbits.

AUTO is a software specialized in the analysis of bifurcations and the continuation of solutions to ODEs, originally developed by Eusebius Doedel with subsequent major contribution by several people. The most important features of the software are now presented, though it is highly recommended to review the official manual [15] for a very detailed explanation of its capabilities and examples of utilization of the software. It can locate branch points and automatically compute solution families that bifurcate from them, it can locate *Hopf bifurcations* and detect its properties, locate folds, find extrema in an objective function and continue it in more parameters.

Nonetheless, it is primarily aimed at the continuation of solutions of ODEs subject to boundary and integral constraints as it will be briefly explained. It can compute families of stable and unstable periodic solutions and compute the Floquet multipliers that determines stability along these families, with starting data generated automatically at Hopf bifurcation points. It can compute folds, period-doubling bifurcations, follow curves of homoclinic orbits. In order to see a comprehensive document about the utilization of AUTO the manual is recommended.

As the main goal of this work isn't to justify the validity of continuation it will only be mentioned that continuation is built upon the *Contraction Theorem* and the *Implicit Function Theorem* (IFT), which together guarantee the existence of a solution family (or branch) $\mathbf{u} = \mathbf{u}(\boldsymbol{\lambda})$, with $\mathbf{u}(\boldsymbol{\lambda}_0) = \mathbf{u}_0$, to the system $\mathbf{G}(\mathbf{u},\boldsymbol{\lambda}) = \mathbf{0}$ when $\mathbf{G}_\mathbf{u}(\mathbf{u}_0, \boldsymbol{\lambda}_0)$ is invertible with bounded inverse and $\mathbf{u}_0$ is an isolated solution of $\mathbf{G}(\mathbf{u},\boldsymbol{\lambda}_0) = \mathbf{0}$. It is also possible to prove that the solution family $\mathbf{u}(\boldsymbol{\lambda})$ is continuously differentiable. Without loss of generality, we will consider the continuation of a one-parameter equation, given that we have the solution $(\mathbf{u}_0, \lambda_0)$ as well as the direction vector $\mathbf{U}_0 = d\mathbf{u}/d\lambda$, and we want to compute a solution $\mathbf{u}_1$ close to $\mathbf{u}_0$. Then, given a small $\Delta\lambda$ such that $\lambda_1 = \lambda_0 + \Delta\lambda$ we just apply Newton's method:

$$\begin{cases} \mathbf{G}_\mathbf{u}(\mathbf{u}_1^{(\nu)},\lambda_1)\Delta\mathbf{u}_1^{\nu} = -\mathbf{G}(\mathbf{u}_1^{\nu},\lambda_1), \\ \mathbf{u}_1^{\nu+1} = \mathbf{u}_1^{\nu} + \Delta\mathbf{u}_1^{\nu}, \quad \nu = 0, 1, 2, \ldots. \end{cases} \tag{6}$$

Where the initial approximation comes from $\mathbf{u}_1^{0} = \mathbf{u}_0 + \Delta\lambda\, \mathbf{U}_0$. The previous is valid if the solution doesn't have folds (singular points in λ), so a new approach is required in general. This approach is called the *Keller's Pseudo-Archlength Continuation*, which basically adds a new equation to ensure that the next solution is close to the last and fixes the distance and direction of this new solution with a new parameter, now the continuation parameter. This method solves now these equations:

$$\begin{cases} \mathbf{G}(\mathbf{u}_1,\lambda_1) = \mathbf{0}, \\ (\mathbf{u}_1 - \mathbf{u}_0)^{*}\dot{\mathbf{u}}_0 + (\lambda_1 - \lambda_0)\dot{\lambda}_0 - \Delta s = 0 \end{cases} \tag{7}$$

which can be solved using Newton's method with the next iterative system:

$$\begin{pmatrix} (\mathbf{G}_\mathbf{u}^1)^{(\nu)} & (\mathbf{G}_\lambda^1)^{(\nu)} \\ \dot{\mathbf{u}}_0{*} & \dot{\lambda}_0 \end{pmatrix} \begin{pmatrix} \Delta\mathbf{u}_1^{(\nu)} \\ \Delta\lambda_1^{(\nu)} \end{pmatrix} = \begin{pmatrix} \mathbf{G}(\mathbf{u}_1^{(\nu)},\lambda_1^{\nu}) \\ (\mathbf{u}_1^{(\nu)} - \mathbf{u}_0)^{*}\dot{\mathbf{u}}_0 + (\lambda_1^{(\nu)} - \lambda_0)\dot{\lambda}_0 - \Delta s \end{pmatrix}, \tag{8}$$

This is the method that AUTO has implemented for the continuation of solutions in algebraic systems, generalized for any number of parameters. Later it will be shown the criticality of this when computing cycles in AUTO, as the first step will be the computation of the equilibria in the CR3BP equations.

In a general way AUTO can solve BVP of the following shape

$$\mathbf{u}'(t) - \mathbf{f}(\mathbf{u}(t),\boldsymbol{\mu},\lambda) = \mathbf{0}, \quad t \in [0,1], \tag{9}$$

where the derivative is with respect to time, and the solution is subjected to the next boundary conditions

$$\mathbf{b}(\mathbf{u}(0),\mathbf{u}(1),\boldsymbol{\mu},\lambda)=\mathbf{0}, \quad \mathbf{b}(\cdot)\in\mathbb{R}^{n_b}, \tag{10}$$

and integral conditions if necessary for the problem

$$\int_0^1 \mathbf{q}(\mathbf{u}(s),\boldsymbol{\mu},\lambda)ds=\mathbf{0}, \quad \mathbf{q}(\cdot)\in\mathbb{R}^{n_q} \tag{11}$$

The goal is to solve the BVP for $\mathbf{u}(\cdot)$ and $\boldsymbol{\mu}$. The parameter λ is the continuation parameter in which the solution $(\mathbf{u},\boldsymbol{\mu})$ is continued. AUTO solves BVP using the method of orthogonal collocation with piecewise polynomials, as it is accurate and allows for adaptive mesh selection. AUTO solves these systems with an efficient method that includes the condensation of parameters by Gauss elimination done in parallel. For a more detailed explanation see [16] and [17].

This is a general approach that can be applied in a wide variety of problems, though it is very straightforward to construct the problem that results in the continuation of periodic solutions (cycles). In order to make the time vary between 0 and 1 the transformation $t \rightarrow t/T$ is applied to the system:

$$\mathbf{u}'(t)=T\mathbf{f}(\mathbf{u}(t),\lambda), \quad \mathbf{u}(\cdot),\mathbf{f}(\cdot)\in\mathbb{R}^n, \quad T,\lambda\in\mathbb{R}. \tag{12}$$

This way, the solution we seek is that which meets

$$\mathbf{u}(0)=\mathbf{u}(1). \tag{13}$$

As the problem has been set out, given a known periodic solution $(\mathbf{u}_{k-1}(\cdot),T_{k-1},\lambda_{k-1})$ (with known period), the period of the next one $(\mathbf{u}_k(\cdot),T_k,\lambda_k)$ in the continuation is an unknown, a free parameter. Then, if it where only for the last two equations, a translated solution from the last one is also a solution: $\mathbf{u}_k(t) = \mathbf{u}_{k-1}(t+\sigma)$. Thus, a phase condition is needed to arrive at the next solution. One option is to use the *Poincaré orthogonality condition*:

$$(\mathbf{u}_k(0)-\mathbf{u}_{k-1}(0))^*\mathbf{u}'_{k-1}(0)=0, \tag{14}$$

though, numerically, a more suitable phase condition is

$$\int_0^1 \mathbf{u}_k(t)^*\mathbf{u}'_{k-1}(t)dt=0. \tag{15}$$

Which is called the integral phase condition [8] and is the one that AUTO implements for the computation of periodic solutions in general. The pseudo-archlength equation is also implemented:

$$\int_0^1 (\mathbf{u}_k(t)-\mathbf{u}_{k-1}(t))^*\dot{\mathbf{u}}_{k-1}(t)dt+(T_k-T_{k-1})^*\dot{T}_{k-1}+(\lambda_k-\lambda_{k-1})\dot{\lambda}_{k-1}=\Delta s. \tag{16}$$

These equations are the ones AUTO solves for the continuation of periodic solutions in general. Besides, during the continuation, AUTO computes the *Floquet multipliers* of the cycles thanks to a special decomposition of the monodromy matrix that arises as a by-product of the Jacobian of the collocation system. It is also possible to compute invariant manifolds using AUTO with the strategy in [18], although this won't be the focus of this paper.

As it has been mentioned AUTO can start to compute a family of periodic solutions from Hopf bifurcations points, which means that the first step will be to compute the equilibria in the CR3BP for our case of study, the Earth-Moon system. This is achieved by performing continuation in the only explicit parameter that the equations of the problem exhibit, the mass parameter μ. We will use that a known solution for the equations exists for $\mu = 0$, as any point of the unit circle is stationary in this case. By executing the problem set up as this, the output of AUTO is a bifurcation diagram with the evolution of the equilibria (U(1) and U(2) are the *(x,y)* coordinates) as μ (PAR(2) in the graph) changes, see figure (1).

Of all those solutions one corresponds to the Earth-Moon system, and that is the one with $\mu = 0.01215$. With this information we are now in disposition to compute periodic orbits that emerge from these points. It is possible to do this thanks to the properties of the equilibrium points in the problem at hand, because some of the eigenvalues are purely imaginary as it was shown in section 2. This means that very close to the them one could linearize the system and express its solution as a combination of the different modes, being some of them sinusoidal and thus periodic. These initial (and analytical) periodic solutions very close to the stationary points can be continued using the complete equations with the method previously shown, which is what AUTO does.

When computing the periodic solutions in conservative systems it is feasible to use the conserved quantity (the Jacobi constant for the CR3BP) as a continuation parameter, which would be a reduction method (adds an equation to decrease by one the degrees of freedom). Instead of doing this, the alternative presented here uses an extra parameter, the unfolding parameter, to increase the dimension of the system in a way that the periodicity only exits when the parameter is 0, so that looking for the cycle is equivalent to looking for a vertical Hopf bifurcation. The system with this parameter becomes:

$$\mathbf{u}'(t)=T\mathbf{f}(\mathbf{u}(t),\lambda)+\alpha\mathbf{d}(\mathbf{u}(t)), \quad \alpha\in\mathbb{R}, \tag{17}$$

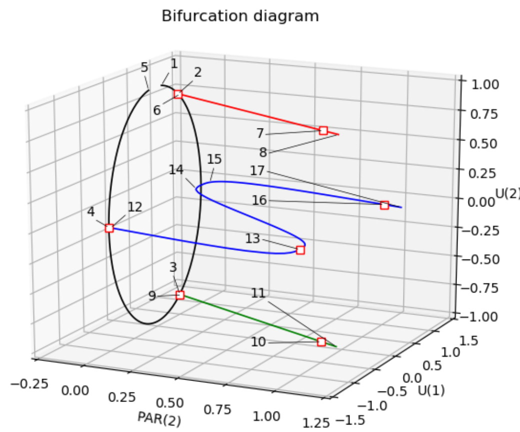

Figure 1: Bifurcation diagram using the built-in tool for
data representation in AUTO

where α is the unfolding parameter and acts as a dissipation (positive or negative). This way it will be $\alpha = 0$ in the periodic solution. Now, by choosing the corresponding Lagrange point as an initial solution, AUTO can perform the continuation of one of the periodic orbit families that grow from it. In particular, we first chose the L1 point, and specify one of the two purely imaginary eigenvalues to continue its corresponding orbital family. The result of this can be seen in figure (2), where the planar Lyapunov L1 family is represented.

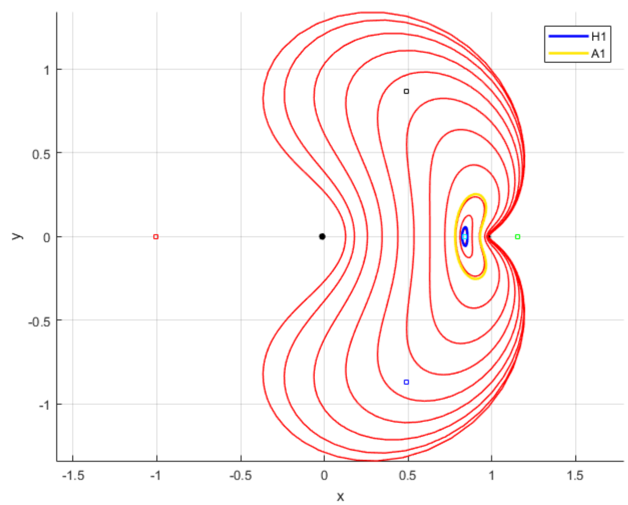

Figure 2: Planar Lyapunov family of Lagrange point L1

There we can see that AUTO not only computes the orbital family required, but it has also detected bifurcations to other families of periodic orbits, such as the halo H1 family and the axial A1 family (the nomenclature followed here is that of [19]). The same can be done for the L2 point, where the bifurcation to the H2 family appears, although no more orbital families that emerge from Lagrange points will be shown in this work for the sake of brevity, see [19] for a more complete representation of the possibilities in this regard. By specifying the H1 and H2 bifurcations as initial solutions in a continuation problem as the previous, AUTO automatically performs the branch switching, computing the orbital families we are after. As it is known, the Halo families have two varieties, the northern and southern type. To perform the branch switching to either of them, the important parameter here is the sign of the continuation step size, as it will define the direction of the continuation. The families calculated here are the northern varieties, which can be seen for H1 and H2 in figure (3).

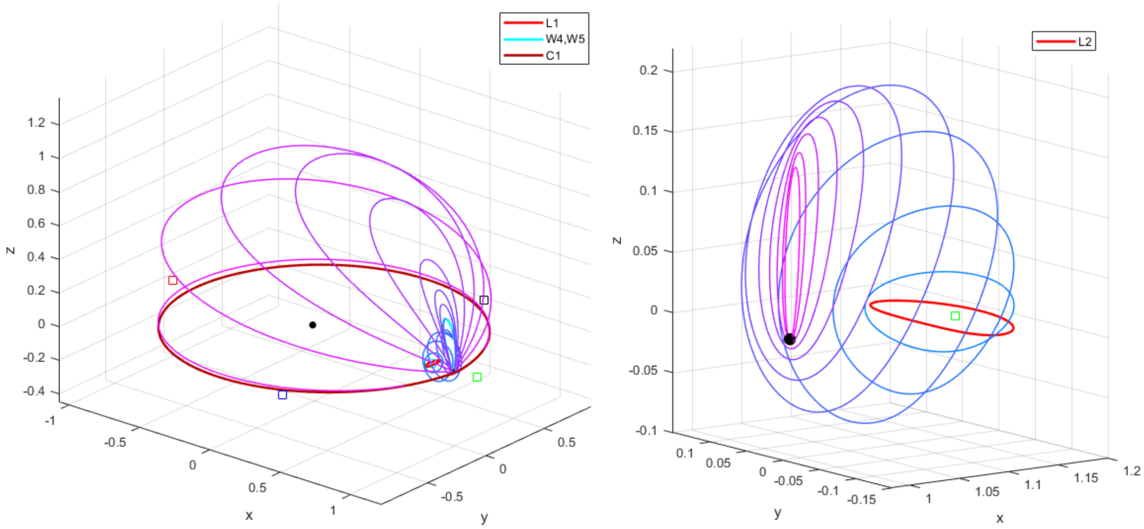

Figure 3: Northern Halo families computed with AUTO

Using the information provided by the Floquet multipliers μ_i it is possible to define the subset of NRHOs as that in which the stability indices $\nu_i = 1/2(\mu_i + 1/\mu_i)$ are within some small bound surrounding ±1 and similar in magnitude, see [12] for a detailed description. Both the H1 and H2 NRHO subset have been represented in figure (4).

Some information such as the period of the orbits is also provided, which allows to study important phenomena like orbit resonances. This is relevant in the design of inhabited stations because of the thermal and power restrictions that come with the eclipsing of the sun. These won't be detailed here for the sake of brevity, it suffices to say that there are resonances in the NRHO subset, with some represented in figure (5) (for more information see [12]).

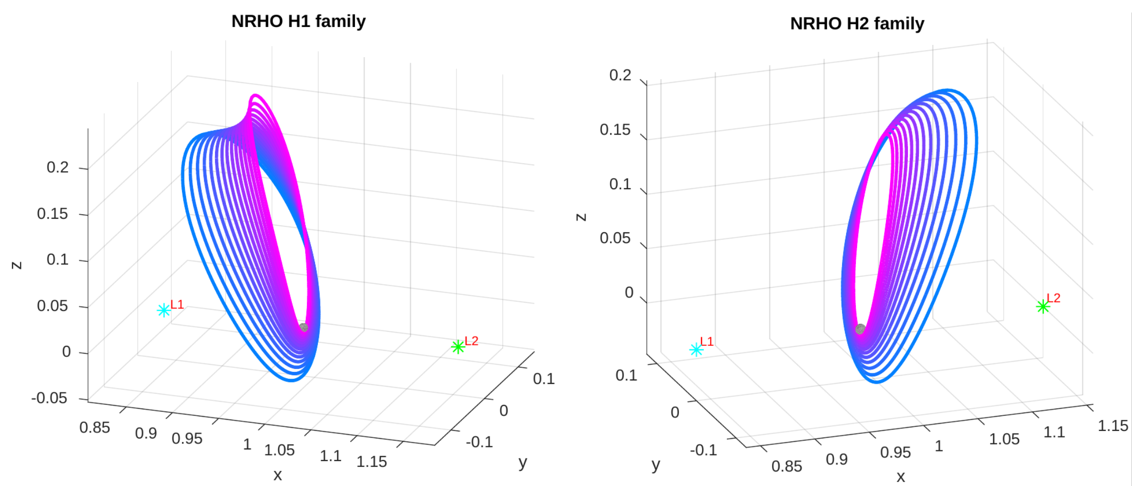

Figure 4: Near Rectilinear Halo Orbits for the H1 and H2 families

IV. Methodology for a complete search in the impulsive maneuver optimization problem

The final goal of this project is to compute the trajectories that arise from the optimization problem of insertion in Halo orbits from LEO. As a means of simplifying the problem in its general complexity we will only consider the optimization of the impulsive maneuvers to achieve the aforementioned trajectories in the CR3BP.

First let us outline the strategy to compute these trajectories, to then face the optimization problem. Given the insertion point in the desired orbit, the trajectory will be computed by backwards time integration of the state before the final free direction impulse (3 variables), so that the end of the integration comes from the monitorization of the distance to Earth in a way that it stops when it reaches a minimum. This can be easily done using the derivative over time, as this function cross zero value when the distance is minimum:

$$f_{event} = \frac{d|\boldsymbol{r}(t) - \boldsymbol{r}_m|}{dt} = \frac{(r_x(t) - r_{mx})\dot{r}_x + (r_y(t) - r_{my})\dot{r}_y + (r_z(t) - r_{mz})\dot{r}_z}{|\boldsymbol{r}(t) - \boldsymbol{r}_m|} \tag{18}$$

This process is done with the *Matlab ODE* suite, as it incorporates an event detection functionality that works as explained. The goal is not only to find any insertion trajectory for a given orbit under study, but to find the most suitable for the mission in the DSG context. In general, we look for the cheapest trajectory in terms of ΔV , but time constraints have to be taken into consideration, as it is a mission with human aspects associated to it. This aspect will later develop in another type of constraint for safety measures. Then, the function to minimize is the cost of the insertion (sum of the norms of each impulse), subjected to at least the constraint that assures that the first collinear impulse is done in the parking LEO. All of this has to be set up as constrained non-linear optimization, as both the constraints and the objective function are of non-linear nature. The topic of constrained non-linear optimization is large enough to deserve its own research work, here it will only be treated from a practical point of view through the problem at hand. It can be expressed in general like this:

$$\min_x f(x) \text{ such that } \begin{cases} c(x) \le 0 \\ ceq(x) = 0 \\ A \cdot x \le b \\ Aeq \cdot x = beq \\ lb \le x \le ub, \end{cases} \tag{19}$$

This problem type can be solved in *Matlab* by using the already implemented non-linear programming solver called *fmincon*, which requires an initial guess and the functions and matrices that define the problem. For a direct 2-impulsive maneuver, given the last impulse (3 variables), the first is also given as that which circularizes at the altitude that it occurs. The problem trajectory is then subjected to the altitude constraint mentioned before, as well as time constraints and any other type of constraint (the inclination of the initial orbit for example). All of this has been implemented in *Matlab* using nested functions for the objective and constraints, so that the numerical integration doesn't have to be repeated.

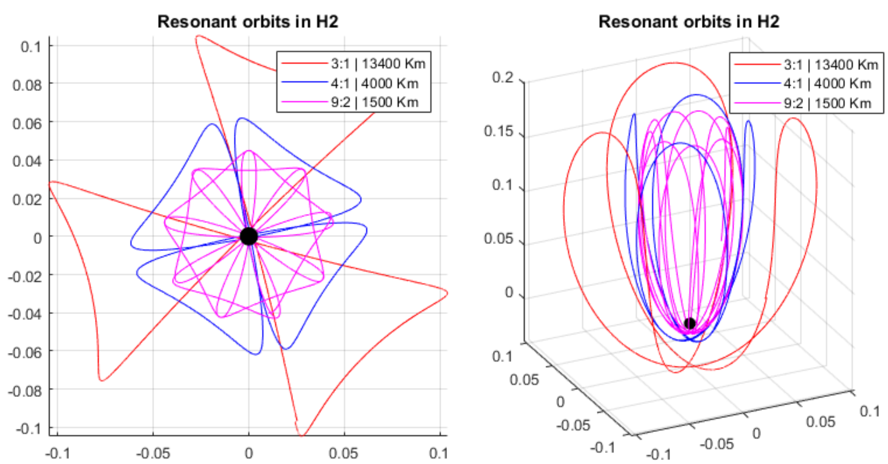

Figure 5: Resonances in the H2 NRHO subset

There is a very important factor that has been mentioned but hasn't been treated with the importance it deserves, and it is the fact that this method requires an initial guess for the solution. This means that in order to compute the optimal solution, a previous knowledge of the problem is required, which could be problematic depending on the problem that is being optimized. A method to provide the initial guess is then needed for this procedure, so that we can have enough confidence in the solution to be the global minimum of the problem. The alternative proposed here uses the powerful tools provided in the *Global Optimization Toolbox* of *Matlab*. *Global Optimization Toolbox* provides functions that can accomplish a more throughout search in problems with multiple maxima or minima, with *MultiStart* [20] being the one used in this work. This function, based on local solvers such as *fmincon*, lets the user choose how many initial search points are used in the process of finding as many local minimum as possible, and uniformly distributes those starting points within bounds. Then, it runs a local solver for all those points (it can be done in parallel) and gives the list of all found solutions as output. Depending on the complexity of the problem and its non-linearity one could tune this value to increase the chance of finding the global optimum. Although this sounds good on paper, the time-consuming aspect of this method could be a problem, as the number of starting points needed will increase with the dimensionality of the problem very rapidly. In the results part of this work some of the explored solutions reach dimension 7, with its corresponding complexity, and the complete search in this solution space is a computational task quite heavy. The idea that has been explored here is to use *MultiStart* not as the solver for the problem, but as a first step to find the shape of all possible solutions. The goal is to make the global search as fast as possible, without worrying about the precision of the found pre-solutions, or if they meet the constraints with low tolerance. The resulting algorithm is then one that provides the approximate shape of all the possible solutions for a given insertion. It has the advantage that now we don't really need to know how the solution will be beforehand, which is of great importance for complex problems. This pre-solutions are then fed to individual *fmincon* solvers with the desired tolerances in order to get the final solutions.

There is an added advantage to this method that can't be overlooked, and that is it does find multiple minimums . In the next section the insertion in different Halo orbits is studied using the methodology that has been developed here, and the aspects of the solutions considered will be both cost and time, as the DSG mission will be of human nature most of the time.

V. Optimization of impulsive insertion in the CR3BP H1 and H2 orbits for the Earth-Moon system

Here the more relevant and interesting results will be shown and commented, starting with the 2-impulsive direct insertions. The methodology followed uses the algorithm explained in last section applied to isolated points along the orbit that are somewhat representative of the behavior around them, such as the top, and bottom of the orbit, the left and right points (extreme points in the *y* coordinate) and the points of zero *z* coordinate. The results are extrapolated to focus in the most interesting areas of each orbit. For example, applying this method to the *H1* NRHOs it is apparent that a direct insertion is cheaper in the low part of the orbit, which is explained by the rotating direction (counter clock-wise as seen from Earth), but it isn't as simple as inserting in the bottom point, see figure (6).

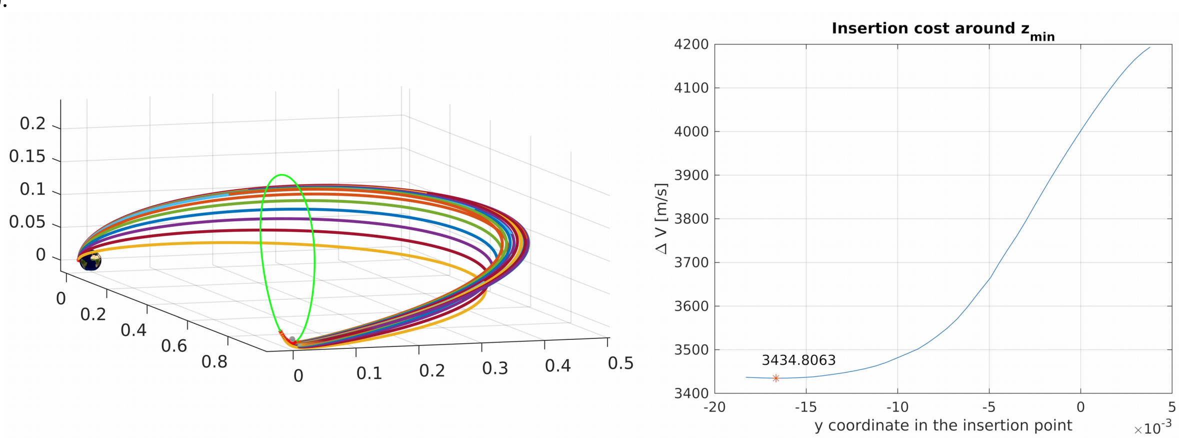

Figure 6: Direct insertion in H1 NRHO along the lower part of the ascending arm

The exploration around the bottom point shows that the gravitational influence of the Moon is enough to displace the optimal insertion point to the ascending arm and improve the cost by more than 500 m/s. The contrary is found to be true in *H2* orbits (clock-wise rotation), where the direct insertion is optimal in the top point of the orbit, see figure (7).

Now the influence of the Moon is not important and the optimal point is placed almost at the top, with very little to gain by going exactly to the point where the costs is minimum. The last figures show the shape of a typical insertion of this kind, as well as the evolution of the cost in the NRHO subset (is higher in smaller orbits, but the change is relatively small). The fact that the Moon plays an important role in the insertion in H1 orbits is a hint on what other types of insertion can be explored. Now, 3-impulsive insertion maneuvers are considered, with the second collinear impulse located at a point of minimum distance to the Moon (using event location just like the first impulse), increasing in one the search space dimension (4 in total now). By using the *MultiStart* approach with this new kind of insertion we can get to study all the casuistry that arises. It has been done for all the relevant points in both the H1 and H2 families, with the most interesting results appearing in H2 (none of the insertions with 3 impulses in H1 where more interesting than the direct one, cost and time-wise). Instead of stopping there, this new type of insertion trajectory can be improved if more optimization variables are unlocked, see figure (8).

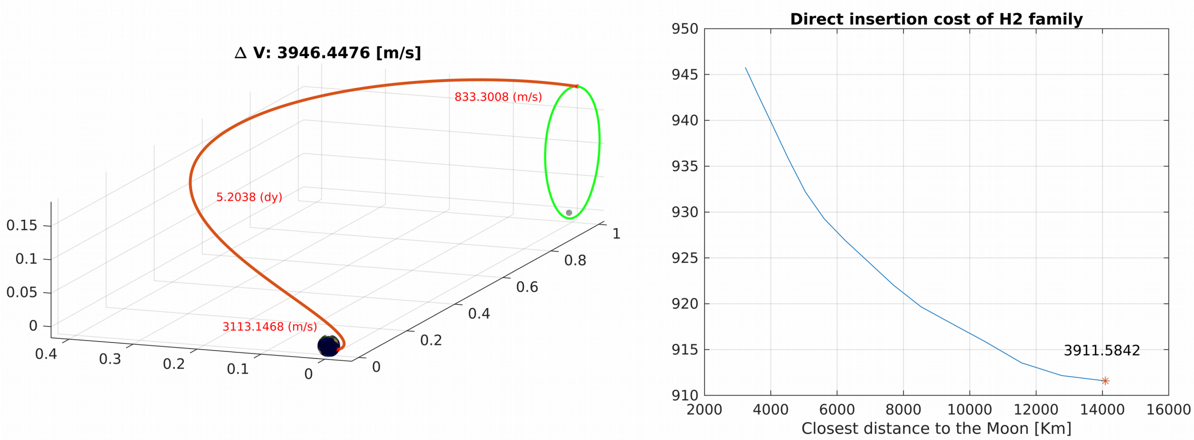

Figure 7: Direct insertion in H2 NRHO (left) and cost along the NRHO subset (right)

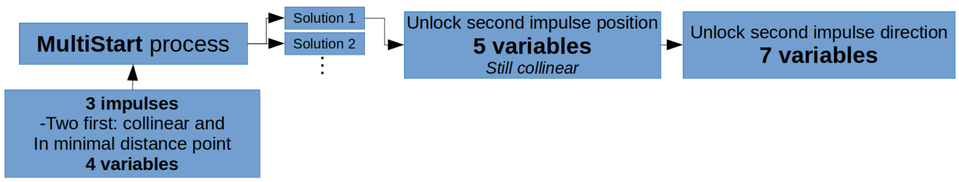

Figure 8: Optimization process to achieve dimension 7 solutions

Using the found trajectory in the 4 variable optimization problem as first guess, we can unlock the integration time in the last stretch of the orbit (so that the second impulse doesn't have to be in a close approach to the Moon) having a 5 variable problem, and the result of that can be further optimized if the second impulse cease to be collinear (7 optimization variables). This has been done around the optimal insertion found in H2 mentioned before, with the results being those in figure (9).

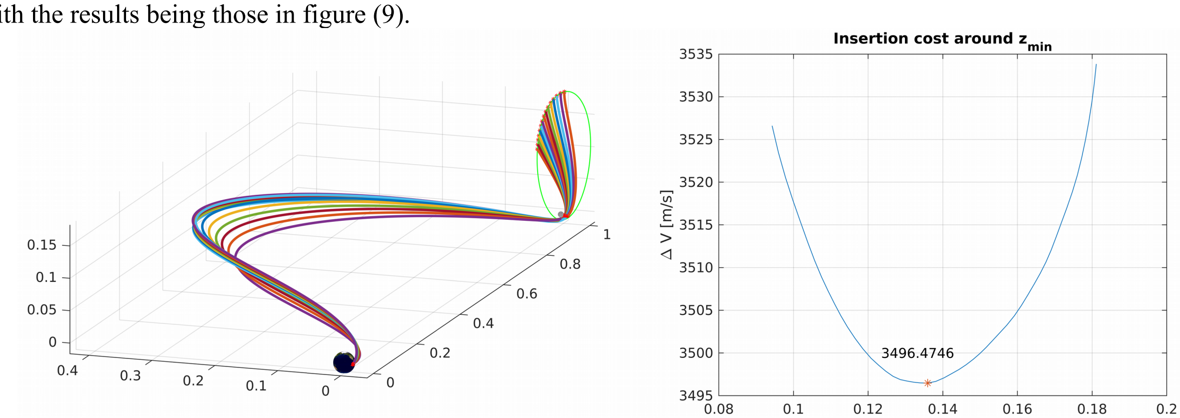

Figure 9: Optimal 3-impulsive insertion in H2 NRHO with 1500 Km in the closest Moon approach

This improves the cost to access H2 orbits with little extra insertion time, and has the added trait of passing very close to the Moon, which could be of use in some missions. For example, consider an hypothetical Moon base in the south pole of the Moon, this trajectory could receive or deploy cargo easily to this facility before arriving to the DSG.

The last result is what comes of considering safety measures for the insertion trajectories, as these are meant to be used in manned missions such as the Apollo ones, which used free-return trajectories in case something went wrong. This last consideration is easily implemented in the methodology as it is just another non-linear constraint (the trajectory that follows the second impulse position has to end in the LEO environment if the maneuver is not performed), and the most interesting result in this regard is the trajectory found in the H1 orbit family, see figure (10).

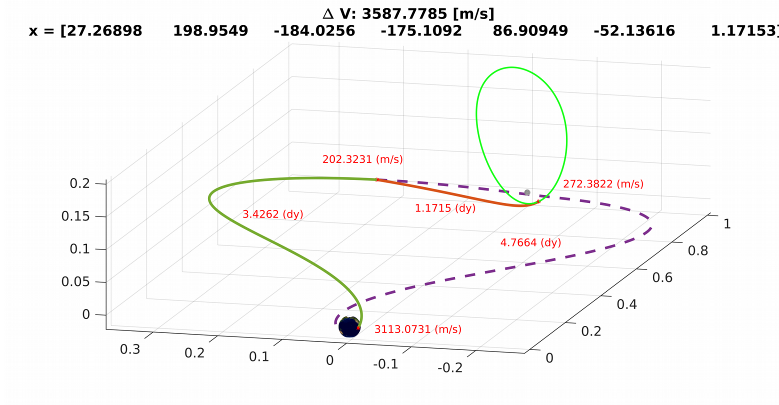

Figure 10: Optimal insertion in terms of cost and time with free-return for a H1 orbit

The cost of this trajectory compared to that of a direct insertion isn't that different, but it has the added advantage of having free-return properties for some time (3 days and a half).

VI. Conclusions and future work

The AUTO software is as an extremely powerful and versatile software that can perform otherwise complex calculations in a very simple way from the user side. This has allowed for a study of the dynamical properties of the cycles that emerge from the equilibria in the CR3BP, with a focus in the Halo orbits that are supposed to be the future home of the *Deep Space Gateway*. On the other hand, this study has the drawback of being a simplification of the real dynamical environment that the DSG will face, which includes the next added difficulties: The Moon orbit has a small eccentricity, which changes the problem from autonomous (easily studied with AUTO) to non-autonomous, as it was revealed with the equations of the ER3BP (see section 2). Although it is technically possible to study time dependent problems with AUTO it adds difficulty in the sense that it would be necessary to include an extra equation for the time (now a state variable), and the study of periodic motion has to be explicitly set as a BVP with periodicity conditions. It could then be considered as a continuation problem with the orbit for e = 0 as initial iteration. The second difficulty is including other perturbations that are potentially critical in long term orbits, such as the Sun's or even Jupiter's. This could be studied separately with higher fidelity models on the orbits that have already been computed to see the consequences of this simplification.

A specific method has been developed in Matlab for the resolution of the non-linear optimization problem of the insertion in Halo orbits from LEO using impulsive maneuvers. Thanks to the results obtained as a consequence, the method gives us a comfortable degree of confidence in the capacity for delivering a complete set of the problem solution types. Regarding the the final results, see section 5, the periodical Halo orbit families of Lagrange points L1 and L2 have been fully explored for the purpose of having an inhabited space station that will serve as a stepping stone for the future missions in the Solar System. The focus has been placed on *Near Rectilinear Halo Orbits*, as they have the most interesting properties for this matter. For the insertion trajectories from LEO, several options have been considered: 2-impulsive maneuvers, that directly connect the LEO environment to the Halo orbit themselves (direct approach), 3-impulsive maneuvers, oriented to use the Moon as an intermediate point (powered lunar fly-by), and finally, using both the direct approach and the 3 impulses method, the safety consideration of having a stretch of the insertion trajectory with free-return properties. Some of the options have being found relevant and interesting for a real application.

The insertion problem has been simplified to use impulsive maneuvers. An extension could be to use the most interesting insertion trajectories and generalize them for an on-off thrust model inspired on known spacecraft such us the *Orion Multi-Purpose Crew Vehicle* (Orion MPCV) of Lockheed Martin or possibly the *Dragon 2* of SpaceX. Higher fidelity physical models (like the ER3BP model) should also be considered to compute and/or test the insertion trajectories in subsequent studies.

The Halo orbits considered in this work are a result of the simplifications of the real environment for the future *Deep Space Gateway;* there are no fully periodical orbits in higher fidelity models. Thus, the station-keeping problem should be addressed. The propulsion module of the DSG itself will use Hall-effect thrusters for station-keeping, and studying the cost of compensating perturbations over time using this kind of propulsion is relevant and necessary.

References

[1] Musielak, Z. E., & Quarles, B. (2014). The three-body problem. *Reports on Progress in Physics*, *77*(6), 065901.

[2] Farquhar, R. W. (1971). The Utilization of Halo Orbits in Advanced Lunar Operations. NASA TN D-6365.

[3] R. Whitley, R. Martinez, G. Condon, J. Williams, and D. Lee. (2016). Cislunar Near Rectilinear Halo Orbit for Human Space Exploration. *AIAA Space,* 2016.

[4] Richardson, D. L. (1980). Analytic Construction of Periodic Orbits About the Collinear Points. *Celestial mechanics*, *22*(3), 241-253.

[5] Thurman, R., & Worfolk, P. A. (1996). The geometry of Halo Orbits in the Circular Restricted Three-Body Problem. *University of Minnesota: Geometry Center Research Report GCG95*.

[6] Asano, Y., Yamada, K., & Jikuya, I. (2015). Approximating Elliptic Halo Orbits Based on the Variation of Constants. *Acta Astronautica*, 113, 169-179.

[7] Nath, P., & Ramanan, R. V. (2016). Precise Halo Orbit Design and Optimal Transfer to Halo Orbits from Earth Using Differential Evolution. *Advances in Space Research*, *57*(1), 202-217.

[8] Osinga, H. M., & Krauskopf, J Galan-Vioque, B. Krauskopf. (2007). Numerical Continuation Methods for Dynamical Systems, 2007.

[9] Dawn Perry Gordon. (2018). Transfers to Earth-Moon L2 Halo Orbits Using Lunar Proximity and Invariant Manifolds. *Purdue University*, August, 2008.

[10] Zeng, H., & Zhang, J. (2016). Design of Impulsive Earth-Moon Halo Transfers: Lunar Proximity and Direct Options. *Astrophysics and Space Science*, *361*(10), 328.

[11] J. Williams, D. E. Lee, R. L. Whitley, K. A. Bokelmann, D. C. Davis, and C. F. Berry. (2017). Targeting Cislunar Near Rectilinear Halo Orbits for Human Space Exploration. *27th AAS/AIAA Space FlightMechanics Meeting*, Feb. 2017.

[12] Zimovan, E. M., Howell, K. C., & Davis, D. C. (2017, May). Near Rectilinear Halo Orbits and Their Application in Cis-Lunar Space. In *3rd IAA Conference on Dynamics and Control of Space Systems, Moscow, Russia*.

[13] Matteo Ceriotti. (2010). Global Optimisation of Multiple Gravity Assist Trajectories. PhD thesis, *University of Glasgow*, 2010.

[14] Schaub, H., & Junkins, J. L. (2003). Analytical Mechanics of Space Systems. *Reston, VA: American Institute of Aeronautics and Astronautics*.

[15] Doedel, E. J., Fairgrieve, T. F., Sandstede, B., Champneys, A. R., Kuznetsov, Y. A., & Wang, X. (2007). AUTO-07P: Continuation and Bifurcation Software for Ordinary Differential Equations.

[16] Ascher, U., Christiansen, J., & Russell, R. D. (1981). Collocation Software for Boundary-Value ODEs. *ACM Transactions on Mathematical Software (TOMS)*, *7*(2), 209-222.

[17] De Boor, C., & Swartz, B. (1973). Collocation at Gaussian Points. *SIAM Journal on Numerical Analysis*, *10*(4), 582-606.

[18] Calleja, R. C., Doedel, E. J., Humphries, A. R., Lemus-Rodríguez, A., & Oldeman, E. B. (2012). Boundary-Value Problem Formulations for Computing Invariant Manifolds and Connecting Orbits in the Circular Restricted Three Body Problem. *Celestial Mechanics and Dynamical Astronomy*, *114*(1-2), 77-106.

[19] Doedel, E. J., Romanov, V. A., Paffenroth, R. C., Keller, H. B., Dichmann, D. J., Galán-Vioque, J., & Vanderbauwhede, A. (2007). Elemental Periodic Orbits Associated with the Libration Points in the Circular Restricted 3-Body Problem. *International Journal of Bifurcation and Chaos, 17*(08), 2625-2677.

[20] Ugray, Z., Lasdon, L., Plummer, J., Glover, F., Kelly, J., & Martí, R. (2007). Scatter Search and Local NLP Solvers: A Multistart Framework for Global Optimization. *INFORMS Journal on Computing*, *19*(3), 328-340.