Evasion maneuvers with double lunar flyby for interplanetary missions

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The purpose of this thesis is to describe and analyze a particular evasion maneuver with double lunar gravity assist (LGA) for interplanetary missions. Lunar flybys are a means to free increase the hyperbolic escape energy (C3) of an escape maneuver for a modest increase in flight time. Two approaches are applied: the first one is the approximate analytical approach; the second one gives the exact numerical solutions, because it takes into account all the main perturbations.

Nomenclature

β	=	Misalignment angle between the velocity directions at infinity and at the escape;
γ_∞	=	Flight path angle at escape;
δ	=	Angle of rotation of the velocity at both flybys;
ϑ	=	Right ascension;
ϕ	=	True anomaly of the asymptote;
φ	=	Declination;
a_J	=	Perturbation due to the Earth asphericity;
a_{lsg}	=	Perturbation due to the luni-solar gravity;
a_P	=	Perturbing acceleration;
a_{srp}	=	Perturbation due to the solar radiation pressure;
ARM	=	Asteroid Redirect Mission;
ARU	=	Asteroid Retrieval and Utilization;
BVP	=	Boundary value problem;
C3	=	Hyperbolic escape energy;
FB1	=	First flyby;
FB2	=	Second flyby;
IVP	=	Initial value problem;
LGA	=	Lunar gravity assist;
ODE	=	Ordinary Differential Equation;
r_M	=	Radius of Moon's circular orbit;
r_P	=	Radius of the circular parking orbit;
RA	=	Right ascension;
S/C	=	Spacecraft;
SOI	=	Sphere of influence;
TA	=	True anomaly;
ΔV	=	Change in velocity.

I. Introduction

THE approximate model, it is easier than the other one because it is based on simplifying assumptions such as: no gravitational influence of Sun and Moon on the spacecraft, no solar radiation pressure, no eccentricity of Moon's orbit (i.e. circular orbit), only Earth's gravitational pull. It is just a preliminary study but extremely helpful because it gives you a macro view of the whole trajectory. Even without going into too much detail, it provides all the most important information in terms of times, positions and velocities. In particular, the solution obtained from this model, is used as the attempt solution for the more detailed analysis. Only short maneuvers, which should be less affected by

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solar perturbation are treated. For preliminary analysis of the interplanetary transfers, is usually adopted the patched conic approximation. The analysis of the heliocentric leg provides us the escape conditions.

In particular, our conditions refer to an Asteroid Redirect Mission (ARM), also known as the Asteroid Retrieval and Utilization (ARU) mission. It is a NASA space mission, proposed in 2013. An ARRM spacecraft (Asteroid Retrieval Robotic Mission) would rendezvous with a large near-Earth asteroid and use robotic arms in order to collect a multiton boulder from its surface and return it to a stable orbit around the Moon or the Earth. This Asteroid Redirect Mission is part of NASA's plan to advance the new technologies and spaceflight experience needed for a human mission to the Martian system in the 2030s. All the solutions presented in the following sections were obtained from these escape conditions as boundary conditions. At the boundary of the Earth's sphere of influence , are known the escape date, positions and escape velocity components of the spacecraft.

Solutions are then compared between the two different approaches and between the different escape conditions. Escape conditions that differ either by the escape date or by the escape velocity.

Comparison between the two different models it is useful in order to verify the approximation committed by the simplest one, taking into consideration that this solution is obtained in less than 1 second (limited computational effort), compared to the much larger effort required to obtain the exact solution.

II. Evasion Maneuvers

In order to escape the gravitational pull of a planet, the spacecraft must travel a hyperbolic trajectory relative to the planet, arriving at its sphere of influence with a relative velocity VŒ (hyperbolic excess velocity) greater than zero. This is a simplification linked to our study. Actually, the truer statement is that S/C must arrive at an infinite distance with a non-zero relative velocity.

The sphere of influence (SOI) is a concept strictly related to the three-body problem. In general it involves: a planet p of mass mp, the Sun s of mass ms and a space vehicle v of mass mv (negligible mass). It is simply a reasonable estimate of the distance beyond which Sun's gravitational attraction dominates that of a planet.

At this distance, our study ends, because beyond this distance it leads from geocentric escape leg to the heliocentric leg.

The radius of a planet's gravitational sphere of influence is calculated as the boundary where the error committed neglecting the Sun in the motion of the spacecraft with respect to the planet, is equal to neglecting the planet in the motion with respect to the Sun. For example, the SOI of the Earth, in the three-body problem with the Sun, is about 925000 km (generally, it is rounded to one million).

The same concepts are applied to the Earth - Moon system, where the sphere of influence of the latter is defined (about 60000 km).

A. Flyby

The introduction of the gravity-assist concept was fundamental to reach the external solar system. So, in this section, it will be briefly exposed the concepts behind the flyby.

A planetary (or lunar) flyby occurs when a spacecraft, entered the SOI of that planet (or the Moon), does not impact or go into orbit around it. The S/C will continue in its hyperbolic trajectory through periapsis and exit the sphere of influence. At the inbound crossing point, velocities assume values as:

$$\bar{V}_{H1} = \bar{V}_1 + \bar{V}_{\infty 1} \tag{1}$$

Similarly, at the outbound crossing, we have

$$\bar{V}_{H2} = \bar{V}_1 + \bar{V}_{\infty 2} \tag{2}$$

The $\overline{\Delta V}$ in the spacecraft's heliocentric velocity is

$$\Delta\bar{V} = \bar{V}_{H2} - \bar{V}_{H1} = (\bar{V}_2 + \bar{V}_{\infty 2}) - (\bar{V}_1 + \bar{V}_{\infty 1}) \tag{3}$$

that is

$$\Delta\bar{V} = \bar{V}_{\infty 2} - \bar{V}_{\infty 1} = \Delta\bar{V}_{\infty} \tag{4}$$

The hyperbolic excess velocity changes its direction but maintains the same magnitude, lie along the asymptotes of the hyperbola. Are therefore inclined at the same angle ϕ to the apse line.

In a trailing-side flyby, the component of $\overline{\Delta V}$ in the direction of the planet's velocity is positive, whereas for leading-side flyby, it is negative. So, a trailing-side flyby results in a increase in the spacecraft's heliocentric speed (or geocentric, for a lunar flyby).
Notice that, if the dimension of the sphere of influence is neglected, the flyby is considered to be an impulsive maneuver during which the heliocentric radius of the spacecraft, which is confined within the planet's sphere of influence, remains fixed at R.
The use of gravity assist maneuvers it is also very helpful to change the orbital parameters of spacecraft's orbit. One of the most important is the inclination, whose variation can be estimated through the simplified model here described. This maneuver allows the S/C velocity vector to rotate relative to the plane of the initial incoming orbit. The angle of rotation depends on both the gravitational capacity of the flyby body and on the location of the point at which the spacecraft enters the body's sphere of activity.

III. Approximate Analytical Approach

The solutions deriving from the approach that will be described in this section, are attempt solutions, necessary to then derive the numerical exact ones.
The analysis is based on a patched-conic approximation that neglects the dimension of Moon's sphere of influence. The trajectory is split into three geocentric legs:
1) the inner leg, from trajectory perigee (usually imposed by the launcher) to the Moon;
2) the intermediate leg, a Moon-to-Moon transfer;
3) the outer leg, from the Moon to the boundary of Earth's sphere of influence (set at 1 million km).

LGA is modeled as an instantaneous relative velocity rotation at Moon's intercept, which separates the geocentric legs. This approach neglects all the external perturbations (as the affect of the Sun for example) and it considers only the affect of the gravitational force exercised by the Earth on the spacecraft (because the S/C remains inside the Earth's sphere of influence).

A. Trajectory Analysis

In this section, the trajectory will be analyzed backwards. In particular, the analysis is carried out with a reference frame based on Moon's osculating orbit: x-axis towards the ascending node of Moon's orbit with respect to Earth's equator, z-axis along angular momentum, y-axis to complete a right-handed reference frame.
It is fundamental that Moon's orbit intercepts spacecraft escape hyperbola at one of the nodes. The escape velocity gives:

$$a_3 = -\frac{1}{V_\infty^2 - 2/r_\infty} \tag{5}$$

Then, after calculating the unit vector along angular momentum:

$$u_h = u_n \times V_\infty \tag{6}$$

with u_h unit vector pointing to the ascending node. The inclination can be obtained from the angular momentum:

$$i_3 = \arccos(u_{\mathrm{h,z}}) \tag{7}$$

Another equation can be obtained from the position of the spacecraft at the second flyby, because the distance from the Earth must be the same for S/C and Moon. This kind of solution is approximated as it assumes first of all the modeling of the lunar orbit around the Earth as circular.

$$r_{\mathrm{M}} = \frac{a_3(1-e_3^2)}{1+e_3\cos(\nu_3)} \tag{8}$$

Moreover, it is important to note that there is a misalignment between the velocity directions at infinity and at the boundary of the sphere of influence, that is

$$\beta = \left(\frac{\pi}{2} - \nu_\infty + \gamma_\infty\right) - \Phi \tag{9}$$

because escape is actually reached at the boundary of the sphere of influence and not at infinity.
At escape, the true anomaly is obtained from

$$r_\infty = \frac{a_3(1-e_3^2)}{1+e_3\cos\nu_\infty} \tag{10}$$

and the flight path angle is

$$\gamma_\infty = \arctan\frac{e_3\sin\nu_\infty}{1+e_3\cos\nu_\infty} \tag{11}$$

β is added to the rotation that must be provided by Moon's flyby.
In this article it is consider the backflip transfer: the Moon is intercepted at points 180 degrees apart, that is, at the intersections of the spacecraft and Moon orbit planes. Moon's orbit is intercepted by spacecraft orbit at ±90 degrees from its perigee. The inclination could be obtained from

$$\frac{1}{a_2} + 2\sqrt{\frac{a_2(1-e_2^2)}{r_M^3}}\cos i_2 = \frac{3}{r_M} - V_\infty^2 \tag{12}$$

which is solved for e_2 given the inclination. The nonlinear system is solved numerically, by introducing the ratio of periapsis radius to Moon's orbit radius

$$\rho = \frac{r_p}{r_M} \tag{13}$$

And so,

$$e_2 = \frac{1-\rho}{\rho - \cos\nu_{fb}} \tag{14}$$

$$a_2 = \frac{\rho}{1-e_2} r_M \tag{15}$$

As before, intercept is on the reference plane and must occur at a node of the spacecraft orbit. Since the magnitude of the relative velocity before and after the flyby must be the same, the following equation can be derived

$$(V_\infty^2 - 3/r_M) + (1-e_1)/r_P = -2\sqrt{r_P/r_M^3}\cos i_1\sqrt{1+e_1} \tag{16}$$

in which the larger solution (plus sign) is for prograde orbits ($i_1 \le \pi/2$), the lower one for retrograde orbits ($i_1 \ge \pi/2$). It is possible to estimate the angle of rotation of the velocity at both flybys

IV. Exact Numerical Solution

In order to obtain spacecraft trajectory, in this document, an indirect method has been used. Indirect methods, are widely used to solve ODE boundary problems (BVP). In particular BVP has been solved by means of shooting procedures.
The shooting methods, solve numerically the BVP problem by reducing it to the initial value problem (IVP). We 'shoot' out trajectories in different directions until we find a trajectory that has the desired boundary value. This method allows you to estimate the initial conditions. To achieve convergence to the solution, it is extremely important to begin with an appropriate tentative solution, as it can be seen in the next sections. In particular, for this document, the tentative solution is obtained via the solution of a similar but easier problem.
The problem has the same number of parameters and of imposed conditions.

Due to numerical convergence problem, the whole trajectory was divided into four phases:

1) From perigee to first flyby;
2) From first flyby to apogee;
3) From apogee to second flyby;
4) From second flyby to escape.

Among which, there are three internal boundaries, where the state variables are discontinuous or constraints are imposed: first periselenium, apogee and second periselenium. Considering internal points, for each one we add a parameter (time) and a condition (relative velocity perpendicular to vector (S/C - Moon) for the periselenium, u = 0 at the apogee). By the division of the trajectory into 4 legs, the shooting method became the multiple shooting method. In the multiple shooting, variables are added after the apogee as parameters and continuity equations as conditions.
The method solves an initial value problem in each of the smaller intervals, and imposes additional matching conditions to form a solution on the whole interval. The division into smaller intervals guarantees improvement on numerical stability over single shooting methods. So, all in all, are solved four BVP transformed into IVP.
The initial values of some of the variables are usually unknown, and the search for the solution results is determined, through an iterative process.
Between the unknowns, there are four times (the final one and the other three due to the internal points), six variables that refer to the apogee in terms of positions and velocity components and five that refer to the initial condition. Moreover, as previously mentioned, to have a continuous solution, at the internal boundaries must be imposed continuity in terms of position and velocity.
Among all the outputs, extremely important will be the comparison between Moon's position and spacecraft's position at both flybys. This aspect will be analyzed in the following section, and one will understand the importance of relative positions.

A. Equations

The spacecraft is modeled as a point with variable mass. Position r, velocity v, and mass m of the spacecraft are the problem state variables, described by differential equation

$$\begin{cases} \frac{d\bar{r}}{dt} = \bar{v} \\ \\ \frac{d\bar{v}}{dt} = -\frac{\mu\bar{r}}{r^3} + \frac{\bar{T}}{m} + \bar{a}_P \\ \\ \frac{dm}{dt} = -\frac{T}{c} \end{cases} \tag{17}$$

The thrust vector T in this document is assumed to be zero. a_p is the perturbing acceleration, given by

$$\bar{a}_P = \bar{a}_J + \bar{a}_{lsg} + \bar{a}_{srp} \tag{18}$$

where: a_j is the perturbation due to the Earth asphericity, a_{lsg} the luni-solar gravity and a_{srp} is the solar radiation pressure.
It is also defined a topocentric reference frame, so, in this frame, the position is easily expressed as $\bar{r} = r\hat{\imath}$, while the velocity as $\bar{v} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$, with u, v, and w being radial, eastward, and northward components, respectively. The scalar state equations can be expressed as:

$$
\begin{cases}
\frac{dr}{dt} = u \\
\frac{d\vartheta}{dt} = \frac{v}{r(\cos\varphi)} \\
\frac{d\phi}{dt} = \frac{w}{r} \\
\frac{du}{dt} = -\frac{\mu}{r^2} + \frac{(v^2+w^2)}{r} + \frac{T_u}{m} + (a_P)_u \\
\frac{dv}{dt} = \frac{(-uv+vw\tan\varphi)}{r} + \frac{T_v}{m} + (a_P)_v \\
\frac{dw}{dt} = \frac{(-uv+v^2\tan\varphi)}{r} + \frac{T_w}{m} + (a_P)_w \\
\frac{dm}{dt} = -\frac{T}{c}
\end{cases}
\tag{19}
$$

In this model,that implements perturbations, the following will be considered:

1) perturbation due to the Earth asphericity;
2) perturbation due to the gravitational attraction of the Sun;
3) perturbation due to the gravitational attraction of the Moon;
4) pressure of solar radiation;
5) eccentricity of Moon's orbit.

V. Approximate Analytical Approach Results

The results reported refer to 5 different escape conditions, at the boundary of the Earth's sphere of influence. Among these, the first 3 (cases 0, 2 and 4) have the same escape date (21/06/2022) but increasing V_∞. However, with regard to the cases 4, 5 and 6 have the same V_∞ but antecedent escape date.

It is important to notice that results, in terms of r, ϑ and φ are those of the Earth in its heliocentric motion:

1) distances are expressed in AU;
2) velocities are the relative ones of the S/C with respect to the Earth;
3) velocities are dimensioned with the velocity of the Earth around the Sun (the reference is heliocentric-ecliptic): 29,78km/s.

As previously said, this conditions refer to an Asteroid Redirect Mission. In particular, the asteroid to be reached is 2008EV5. This asteroid, of the Aten group (a group of asteroids, whose orbit brings them into proximity with Earth, with a semi-major axis of less than 1 AU and with an high eccentricity), was first observed on 4 March 2008. EV5 rotates retrograde and its overall shape is a 400 ± 50 m oblate spheroid, so is defined as a sub-kilometer asteroid. It is a near-Earth object and potentially hazardous. On 23 December 2008, 2008EV5 made a close approach to Earth at a distance of 3.2 million km (0,022 AU), its closest until 2169. It has a semi-major axis of 0,958242 AU, an eccentricity of 0,083401 and an inclination of 7,437 degrees.

Table 1: Escape conditions

	Case 0	Case 2	Case 4	Case 5	Case 6
r	1,016193	1,016193	1,016193	1,016111	1,016023
ϑ	4,702943	4,70294	4,702943	4,683577	4,664207
φ	4,843999E-5	4,84399E-5	4,84399E-5	4,84882E-5	4,85183E-5
u	-3,6668E-3	-3,9547E-3	-4,2507E-3	-4,53042E-3	-4,8141E-3
v	1,66059E-2	1,69436E-2	1,72617E-2	1,70068E-2	1,67514E-2
w	-4,0197E-2	-4,1853E-2	-4,3513E-2	-4,3585E-2	-4,3653E-2
Escape date	21/ 6/2022	21/ 6/2022	21/ 6/2022	19/ 6/2022	18/ 6/2022
$\mathbf{V}_\infty$	1,3	1,35	1,4	1,4	1,4

A. Case 0

The following tables (also of the following sections), express the positions of spacecraft and moon in spherical coordinates (radius r, longitude ϑ and latitude φ), at distinct points of the trajectory like the two flybys, the departure (from a LEO orbit) and the escape.
The magnitude of the radii is adimensionalized with respect to the terrestrial radius (6378,14 km), while the velocities with respect to the corresponding circular speed. Instead, as the name implies, the radius of periselenium is measured with respect to the centre of the Moon and again, the adimensionalized value is calculated as the other radii. The velocity has a radial component u (i.e. towards the Zenit), one in the east direction v and a northward w. The classical orbital parameters were used (semi-major axis, eccentricity, inclination, longitude of the ascending node, argument of periapsis and true anomaly).

Table 2: Comparison of S/C-Moon positions at the two flybys between exact and approximate solution

	Parameter		Exact	Approximated
FB1	r	S/C	64,30886	60,56857
		Moon	64,02390	60,33634
	ϑ	S/C	1,26703	1,34469
		Moon	1,26452	1,34238
	φ	S/C	0,42132	0,43526
		Moon	0,42692	0,44035
FB2	r	S/C	56,26755	60,04843
		Moon	56,59549	60,33634
	ϑ	S/C	4,41875	4,48059
		Moon	4,42317	4,48397
	φ	S/C	-0,42654	-0,43773
		Moon	-0,43008	-0,44035

The distances at which the flyby occur are extremely different given that only in the exact numerical solution it is considered Moon's eccentricity.

Table 3: Comparison of relative positions at the two flybys between exact and approximate solution

	Parameter	Exact	Approximated
FB1	ϑ_{SC}-ϑ_M	0,00251234	0,002312319
	φ_{SC}-φ_M	-0,00560428	-0,005092084
	$r_{periselenium}$	0,48176806	0,4059
	$r_{periselenium}$ [km]	3072,78235	2588,885524
FB2	ϑ_{SC}-ϑ_M	-0,004428128	-0,003376191
	φ_{SC}-φ_M	0,00353543	0,002615465
	$r_{periselenium}$	0,446113524	0,3762
	$r_{periselenium}$ [km]	2845,37283	2399,454876

You can appreciate that, even changing the model, relative positions maintain the same sign. Remember that Moon's radius is equal to 1737 km. During the first flyby the spacecraft is located at an altitude of only 1335 km from the lunar surface, while, at the second flyby, the altitude is lower and is about 1108 km.

Table 4: Comparison of S/C positions between exact and approximate solution

	Parameter	Exact	Approximated
	r	1,03020	1,03020
Perigee	ϑ	-1,60680	-1,54073
	φ	-0,47	-0,36676
	r	64,30886	60,56857
FB1	ϑ	1,26703	1,34469
	φ	0,42132	0,43526
	r	56,26755	60,04843
FB2	ϑ	4,418746267	4,48059
	φ	-0.4265	-0,43773
	r	156.7856	156,78561
Escape	ϑ	0,0314	-0,41
	φ	-1,3333	-1,32022

At both flybys, the values of ϑ calculated through the approximated model are greater than those calculated with the exact model. As visible in the figure below (Fig. 1), where the blue line refers to the Moon's orbit in the case of the most complex solution while, the red one it is relative to the simplest one. Furthermore, it can be noted that the the escape conditions have a strong component outside the ecliptic plane. The φ angle is extremely high in module and in particular is negative. So, according to these data, the escape happens with a latitude close enough to -90 degrees.
The most important orbital parameters for our study are certainly the semi-major axis, the eccentricity and the inclination. All these 3 values are about constant from one model to another. Moreover, only the final leg turns out to be a hyperbolic orbit (for obvious reasons), while the previous two are elliptic.
But, it can be seen that the semi-major axis from the first to the second leg decreases, with the consequent reduction of energy. The importance of the first flyby is to achieve the right inclination (from 0,5 to 1,78 radians) o then the right trajectory. The second flyby, then is the one that significantly increases the energy.
Energy values varies as said before, and as can be clearly seen from the table.
In particular, all the following figures have the equatorial plane as their XY plane and the reference is centered in the centre of mass of the Earth. Therefore, a geocentric reference system is being used. In fact, it can be seen that Moon's orbit is inclined with respect to the ecliptic.

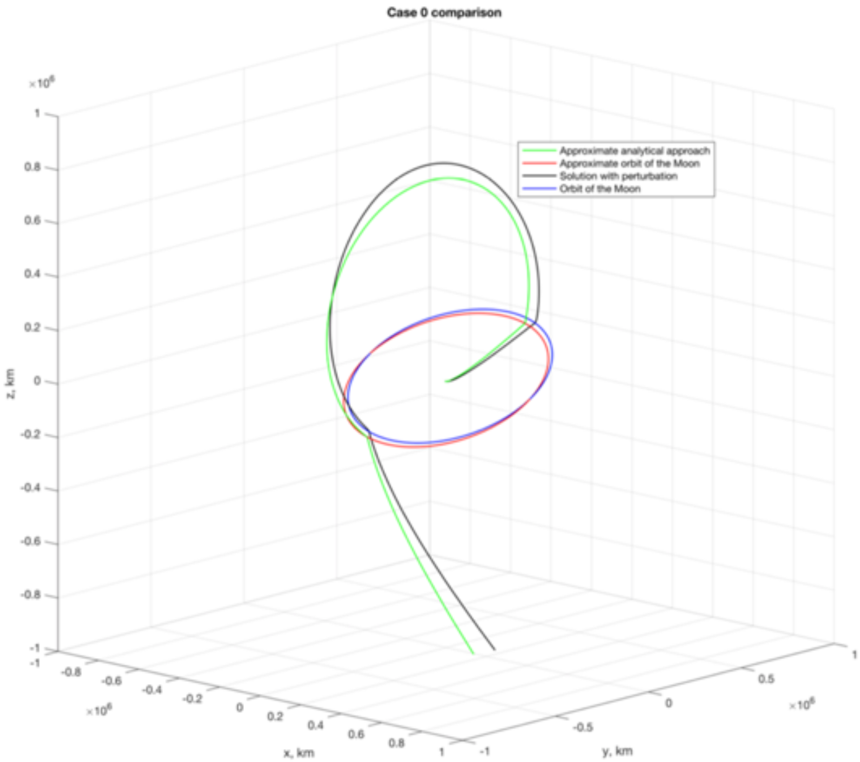

Figure 1. Comparison between exact and approximate solution, Case 0

B. Comparision Between Solutions With The Same Escape Date

Table 5 : Comparison of S/C-Moon positions at the two flybys between solutions with the same escape date

	Parameter		Case 0	Case 2	Case 4
FB1	r	S/C	60,56857	60,56831	60,56609
		Moon	60,33634	60,33633	60,33633
	ϑ	S/C	1,34469	1,41183	1,47594
		Moon	1,34238	1,40951	1,47361
	φ	S/C	0,43526	0,44515	0,45290
		Moon	0,44035	0,44994	0,45738
FB2	r	S/C	60,04843	60,07686	60,10411
		Moon	60,33634	60,33633	60,33633
	ϑ	S/C	4,48059	4,54773	4,61188
		Moon	4,48397	4,55111	4,61520
	φ	S/C	-0,43773	-0,44747	-0,45508
		Moon	-0,44035	-0,44994	-0,45738

The first thing you notice, as already mentioned, is that with the approximate solutions the distances of the Moon from the Earth at both flybys is the same (about 384400 km, that is, the average Moon-Earth distance) because Moon's orbit is circular. Whereas, these distances, due to the eccentricity of Moon's orbit, vary a lot, from a minimum value of about 360000 km to a maximum value of 410000 km.
Another aspect to consider are the angular positions (both of the S/C and of the Moon) that vary from one case to another,

• ϑ: as can be seen, at both flybys, grows from case 0 to 4 (this indicates that the two flybys occur in successive moments with respect to case 0);
• φ: grows from case 0 to 4 at the first flyby (when the Moon is intercepted it is at a higher latitude, compared to the geocentric reference), while at the second one decreases;
• The variation of ϑ is one order of magnitude greater than that of φ.

Table 6 : Comparison of relative positions at the two flybys between solutions with the same escape date

	Parameter	Case 0	Case 2	Case 4
FB1	ϑ_{SC}-ϑ_M	0,002312319	0,00232	0,00233
	φ_{SC}-φ_M	-0,005092084	-0,00479	-0,00448
	$r_{periselenium}$	0,4059	0,39204	0,37690
	$r_{periselenium}$ [km]	2588,885524	2500,46514	2403,91321
FB2	ϑ_{SC}-ϑ_M	-0,003376191	-0,00338	-0,00333
	φ_{SC}-φ_M	0,002615465	0,00246	0,00230
	$r_{periselenium}$	0,3762	0,35054	0,32481
	$r_{periselenium}$ [km]	2399,454876	2235,78593	2071,68764

For what concern about the relative angular positions between S/C and Moon at both flybys, they can be summarized as:
• At the first flyby the spacecraft is later (ϑ) and lower (φ) than the Moon;
• At the second flyby, contrary to what happens to the first one, the spacecraft is more backward (ϑ) and higher (φ) than the Moon.
The knowledge of these mutual positions was fundamental in order to calculate the exact numerical solution. Only with this exact angular configuration it was possible to obtain the searched solution.

Table 7 : Comparison of orbital parameters between solutions with the same escape date, part 1

		a	e	incl
Perigee-FB1	Case 0	300,64615	0,99657	0,50116
	Case 2	640,87874	0,99839	0,49922
	Case 4	-2938,48377	1,00035	0,50305
FB1-FB2	Case 0	88,80794	0,56617	1,78919
	Case 2	88,80536	0,56616	1,80714
	Case 4	88,79413	0,56612	1,83020
FB2-ESCAPE	Case 0	-70,00101	1,80220	1,32066
	Case 2	-60,95783	1,92457	1,29743
	Case 4	-53,74224	2,05185	1,27611

Table 8 : Comparison of orbital parameters between solutions with the same escape date, part 2

		RA	w	TA
Perigee-FB1	Case 0	5,51959	5,44057	6,28318
	Case 2	5,63918	5,39956	6,28318
	Case 4	5,72239	5,38858	6,28318
FB1-FB2	Case 0	1,44715	5,16438	1,57066
	Case 2	1,52610	5,17628	1,57070
	Case 4	1,60460	5,18687	1,57080
FB2-ESCAPE	Case 0	1,22166	3,12853	1,56878
	Case 2	1,27369	3,15889	1,54414
	Case 4	1,32361	3,18454	1,52188

Talking about orbital parameters, the biggest difference can be seen in the calculation of the semi-major axis for the Perigee-FB1 leg. The main problem you have is in case 4, where the solution gives a negative semi-major axis (hyperbolic orbit), while in the other cases a positive value is obtained (which refers to an elliptical orbit). According to what has just been said, the eccentricity is greater than or less than 1 with minimal differences from the unit.
One of the most important parameters, the inclination between the two flybys, remains almost unchanged (the maximum difference is two degrees). As it happens in the other 2 legs. It can be noted that the initial inclination, about 0,5 radians (approximately 28 degrees), it's the maximum angle between Moon's orbit and Earth's equator.

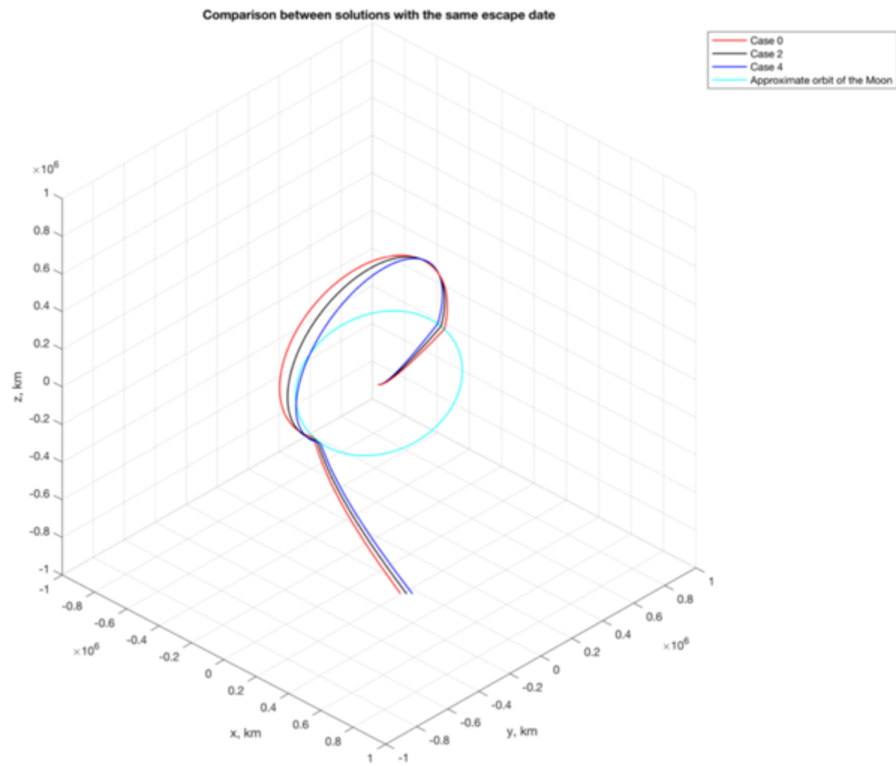

Figure 2. Comparison between solutions with the same escape date

C. Comparision Between Solutions With The Same Hyperbolic Escape Velocity

Table 9 : Comparison of S/C-Moon positions at the two flybys between solutions with the same V_∞

	Parameter		Case 0	Case 4	Case 5	Case 6
FB1	r	S/C	60,56857	60,56609	60,64891	60,84856
		Moon	60,33634	60,33633	60,43736	60,64968
	ϑ	S/C	1,34469	1,47594	1,03329	0,66144
		Moon	1,34238	1,47361	1,03033	0,65828
	φ	S/C	0,43526	0,45290	0,36798	0,23709
		Moon	0,44035	0,45738	0,37159	0,23950
FB2	r	S/C	60,04843	60,10411	60,21413	60,46568
		Moon	60,33634	60,33633	60,43736	60,64968
	ϑ	S/C	4,48059	4,61188	4,16885	3,79767
		Moon	4,48397	4,61520	4,17192	3,79988
	φ	S/C	-0,43773	-0,45508	-0,36783	-0,23490
		Moon	-0,44035	-0,45738	-0,37159	-0,23950

Contrary to what happened to the escape with the same date, in this case the value of ϑ at both flybys and φ at the first one, decreases significantly by anticipating the escape. Recalling that, the escape of case 6 occurs before the one of case 5, which in turn has earlier escape with respect to case 4. At the second flyby, the value of φ, in antithesis with the previous study, grows.

Table 10: Comparison of relative positions at the two flybys between solutions with the same V_∞

	Parameter	Case 0	Case 4	Case 5	Case 6
FB1	ϑ_{SC}-ϑ_M	0,002312319	0,00233	0,00296	0,00316
	φ_{SC}-φ_M	-0,005092084	-0,00448	-0,00361	-0,00241
	$r_{periselenium}$	0,4059	0,37690	0,34707	0,30932
	$r_{periselenium}$ [km]	2588,885524	2403,91321	2213,68900	1972,89432
FB2	ϑ_{SC}-ϑ_M	-0,003376191	-0,00333	-0,00307	-0,00221
	φ_{SC}-φ_M	0,002615465	0,00230	0,00376	0,00460
	$r_{periselenium}$	0,3762	0,32481	0,36216	0,35831
	$r_{periselenium}$ [km]	2399,454876	2071,68764	2309,91443	2285,33479

For what concern about the relative angular positions between S/C and Moon at both flybys, they vary in magnitude but not in sign between all the 5 cases.
So, the reciprocal positions are fixed to obtain this type of trajectory.

Table 11: Comparison of orbital parameters between solutions with the same V_∞, part 1

		a	e	incl
Perigee-FB1	Case 0	300,64615	0,99657	0,50116
	Case 4	-2938,48377	1,00035	0,50305
	Case 5	-472,11528	1,00218	0,50501
	Case 6	-136,47187	1,00754	0,50274
FB1-FB2	Case 0	88,80794	0,56617	1,78919
	Case 4	88,79413	0,56612	1,83020
	Case 5	88,95768	0,56620	2,08356
	Case 6	89,26654	0,566157	2,40302
FB2-ESCAPE	Case 0	-70,00101	1,80220	1,32066
	Case 4	-53,74224	2,05185	1,27611
	Case 5	-53,74761	2,09924	1,49193
	Case 6	-53,74261	2,121569	1,70933

Table 12: Comparison of orbital parameters between solutions with the same V_∞, part 2

		RA	w	TA
Perigee-FB1	Case 0	5,51959	5,44057	6,28318
	Case 4	5,72239	5,38858	6,28318
	Case 5	4,95423	5,70438	0,00001
	Case 6	4,26003	6,05695	6,28318
FB1-FB2	Case 0	1,44715	5,16438	1,57066
	Case 4	1,60460	5,18687	1,57080
	Case 5	1,25153	5,14225	1,57074
	Case 6	0,92981	5,07259	1,57068
FB2-ESCAPE	Case 0	1,22166	3,12853	1,56878
	Case 4	1,32361	3,18454	1,52188
	Case 5	0,99950	3,25602	1,49074
	Case 6	0,69236	3,24931	1,47635

The semi-major axis of the first leg, in the last 3 cases, remains negative while the eccentricity grows a little. In the same way, even the inclination of the second leg grows a lot, with a margin of about 35 degrees.

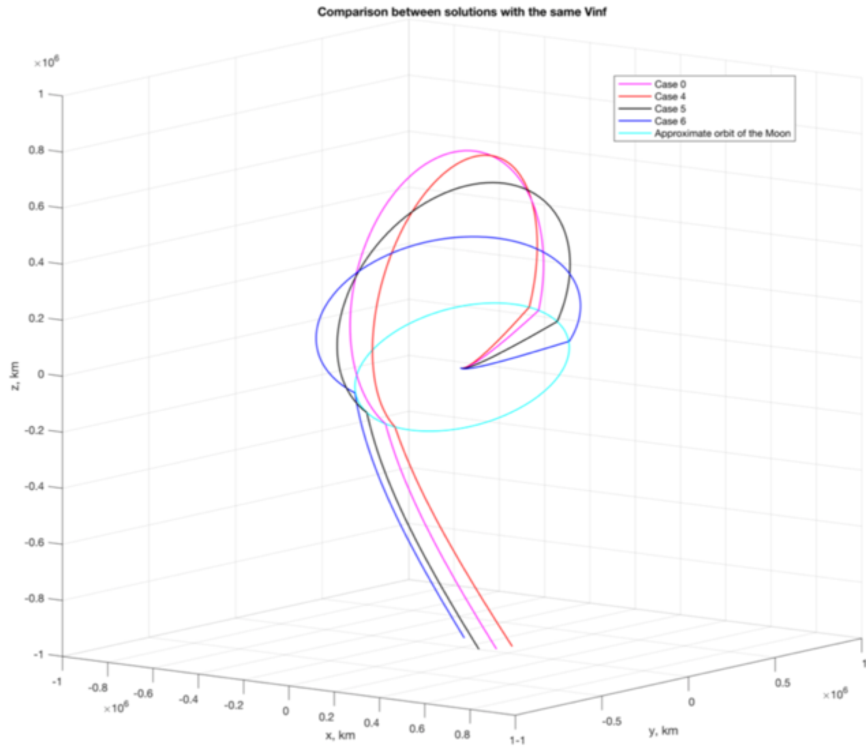

Figure 3. Comparison between solutions with the same V∞

Moreover, it can be notice that changing the escape date (cases 5 and 6) varies considerably the Perigee-Moon leg: in fact, in cases 0 2 and 4 (considering the solution which takes into account perturbations) this leg is elliptical while, in the other 2 cases, orbit is hyperbolic.
In this case, the final energy maintains the same value that is still greater than the reference case (case 0), because, compared to it, the final leg has greater semi-major axis
as well as V_{∞}.

VI. Exact Numerical Solutions Results

A. Comparision Between Solutions With The Same Escape Date

Table 13: Mass and times as the V_∞ varies

Case	0	2	4
Mass [kg]	10318,06402	10310,76261	10301,86671
Mass correct [kg]	9768,06402	9760,76261	9751,86671
V∞	1,3	1,35	1,4
T1	255,87464	254,22257	252,00386
T2	2457,49223	2452,95069	2449,59207
T3	4713,07637	4704,69311	4696,55410
T4	5491,59696	5454,88856	5420,41797
Date of FB1	30/4/2022	1/5/2022	1/5/2022
Date of apogee	3/5/2022	3/5/2022	3/5/2022
Date of FB2	13/6/2022	14/6/2022	14/6/2022
Date of escape	21/6/2022	21/6/2022	21/6/2022

It can be noticed immediately that, increasing V_∞ the mass at the escape is a bit lower. In the same way varies the time needed to reach the boundary of the sphere of influence.
In particular, most of the time is occupied from the first flyby to the second one (about the 80% of the total flight time).

Table 14: Comparison of S/C-Moon positions at the two flybys

		Case	**0**	**2**	**4**
FB1	r	S/C	64,30885577	64,37878	64,42534
		Moon	64,02390329	64,09538	64,14464
	ϑ	S/C	1,26702922	1,33862	1,40507
		Moon	1,26451688	1,33617	1,40266
	φ	S/C	0,42131895	0,43392	0,44380
		Moon	0,42692323	0,43937	0,44904
	$r_{periselenium}$		0,48176806	0,47267	0,46059
	$r_{periselenium}$ [km]		3072,78234	3014,73757	2937,70780
FB2	r	S/C	56,26755091	56,24613	56,24091
		Moon	56,59548797	56,54160	56,50561
	ϑ	S/C	4,418746267	4,49328	4,56346
		Moon	4,423174	4,49765	4,56771
	φ	S/C	-0,4265	-0,43921	-0,44906
		Moon	-0,43007	-0,44245	-0,45203
	$r_{periselenium}$		0,4461135243	0,41293	0,38034
	$r_{periselenium}$ [km]		2845,37286	2633,72663	2425,86123

The distances of the spacecraft at both flybys decreases as V_∞ increases, in particular that relating to the second one.

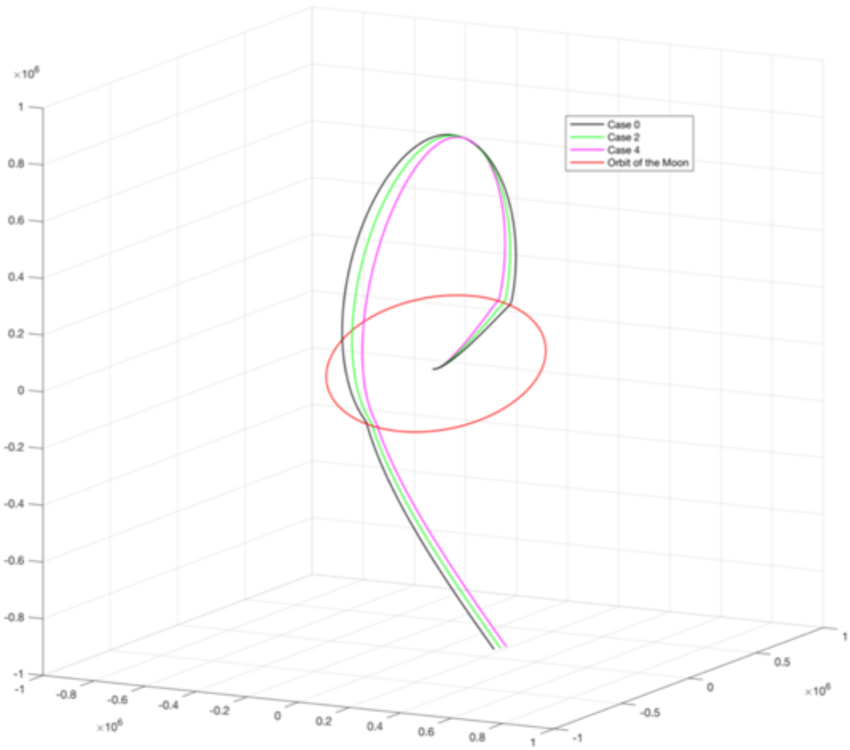

Figure 4. Comparison between solutions as the V∞ varies, 3D view

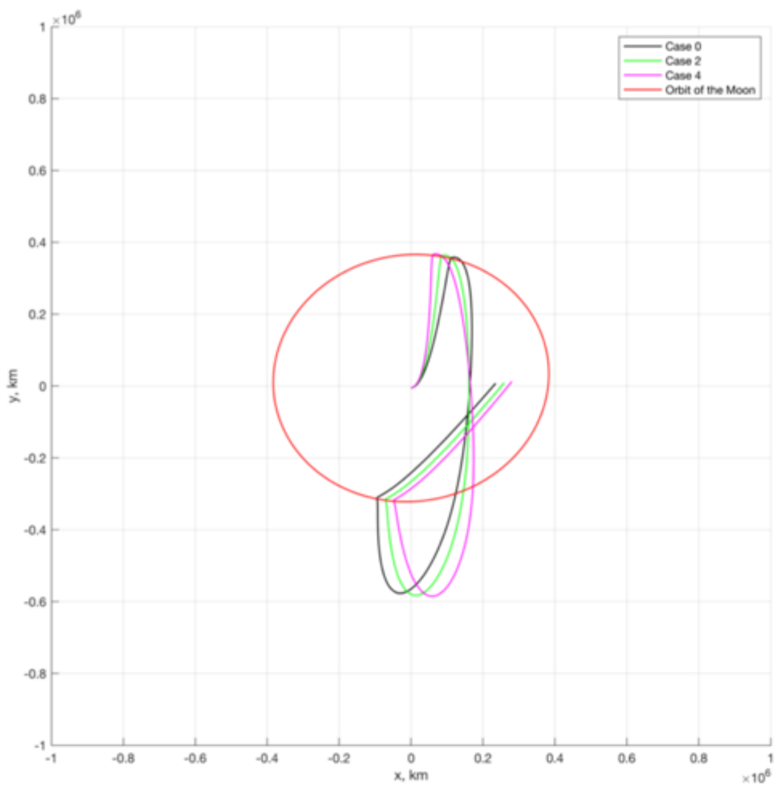

Figure 5. Comparison between solutions as the V∞ varies, XY view

As shown in the Fig. 4 and 5 the 3 trajectories are very similar to each other, but slightly rotated forward with respect to each other: if the values of φ vary a little, ϑ grows much more.
In this view (Fig. 4, in which it is possible to have a 3-dimensional view of the entire trajectory), you can realize that the Moon-Moon leg is obtained through a retrograde orbit (inclination greater than 90 degrees).

B. Comparision Between Solutions With The Same Hyperbolic Escape Velocity

Table 15: Mass and times as the date of escape varies

Case	0	4	5	6
Mass [kg]	10318,06402	10301,86671	10246,20564	10163,21690
Mass correct [kg]	9768,06402	9751,86671	9696,20564	9613,21690
$\mathbf{V}_\infty$	1,3	1,4	1,4	1,4
T1	255,87464	252,00386	236,59710	216,06620
T2	2457,49223	2449,59207	2466,66770	2455,65680
T2	4713,07637	4696,55410	4742,98600	4801,47600
T3	5491,59696	5420,41797	5505,84100	5592,87430
Date of FB1	30/4/2022	1/5/2022	29/4/2022	27/4/2022
Date of apogee	3/5/2022	3/5/2022	1/5/2022	29/4/2022
Date of FB2	13/6/2022	14/6/2022	12/ 6/2022	11/ 6/2022
Date of escape	21/6/2022	21/6/2022	19/ 6/2022	18/ 6/2022

It can be noted that the distance of the Moon from the Earth at flybys is different: it is farther at the first one, and overall the difference of radius is about 50000 km. This happens because it is considered the eccentricity of Moon's orbit, which did not happen in the approximate approach.

Table 16: Comparison of S/C-Moon positions at the two flybys

		Case	0	4	5	6
FB1	r	S/C	64,30885577	64,42534	64,11500	63,18250
		Moon	64,02390329	64,14464	63,86302	62,95801
	ϑ	S/C	1,26702922	1,40507	0,97670	0,59560
		Moon	1,26451688	1,40266	0,97355	0,59234
	φ	S/C	0,42131895	0,44380	0,35140	0,20930
		Moon	0,42692323	0,44904	0,35484	0,21108
	$r_{periselenium}$		0,48176806	0,46059	0,38434	0,32267
	$r_{periselenium}$ [km]		3072,78234	2937,70780	2451,39280	2058,06496
FB2	r	S/C	56,26755091	56,24091	56,56920	57,56510
		Moon	56,59548797	56,50561	56,80579	57,74568
	ϑ	S/C	4,418746267	4,56346	4,13269	3,77159
		Moon	4,423174	4,56771	4,13647	3,77433
	φ	S/C	-0,4265	-0,44906	-0,35670	-0,22320
		Moon	-0,43007	-0,45203	-0,36128	-0,22865
	$r_{periselenium}$		0,4461135243 3	0,38034	0,40694	0,39259
	$r_{periselenium}$ [km]		2845,37286	2425,86123	2595,54766	2504,02244

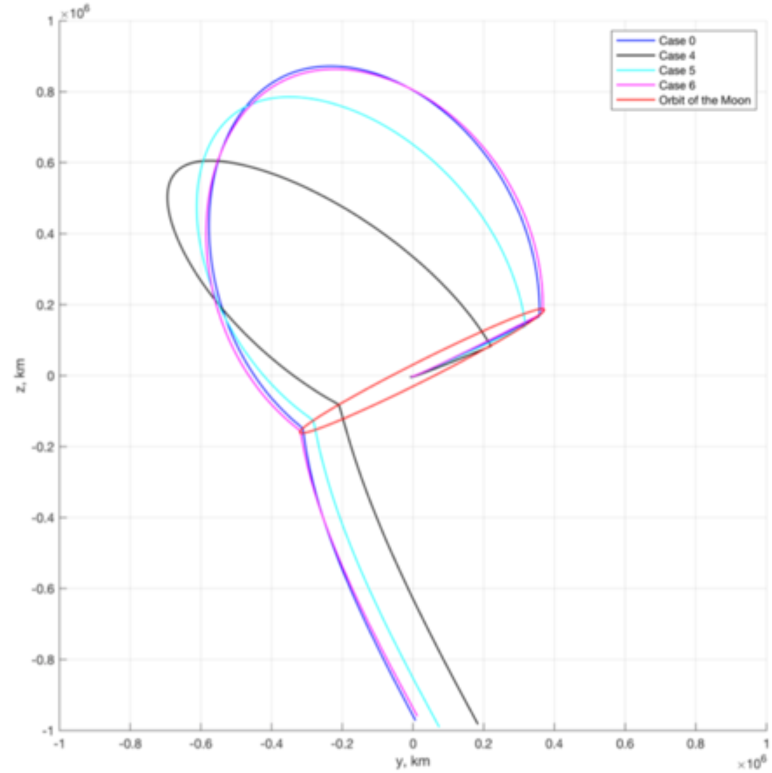

Figure 6. Comparison between solutions as the escape date varies, YZ view

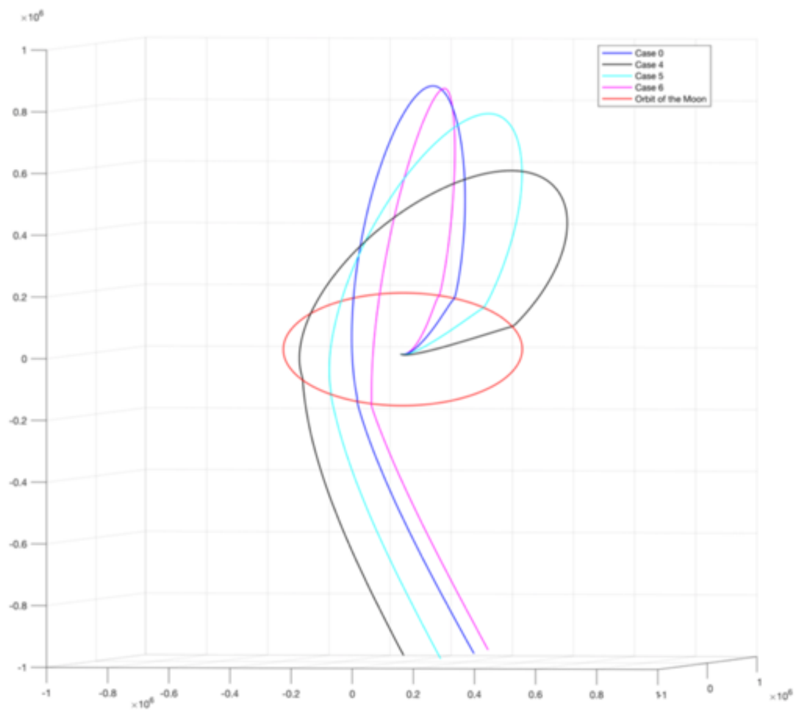

Figure 7. Comparison between solutions as the escape date varies, 3D view

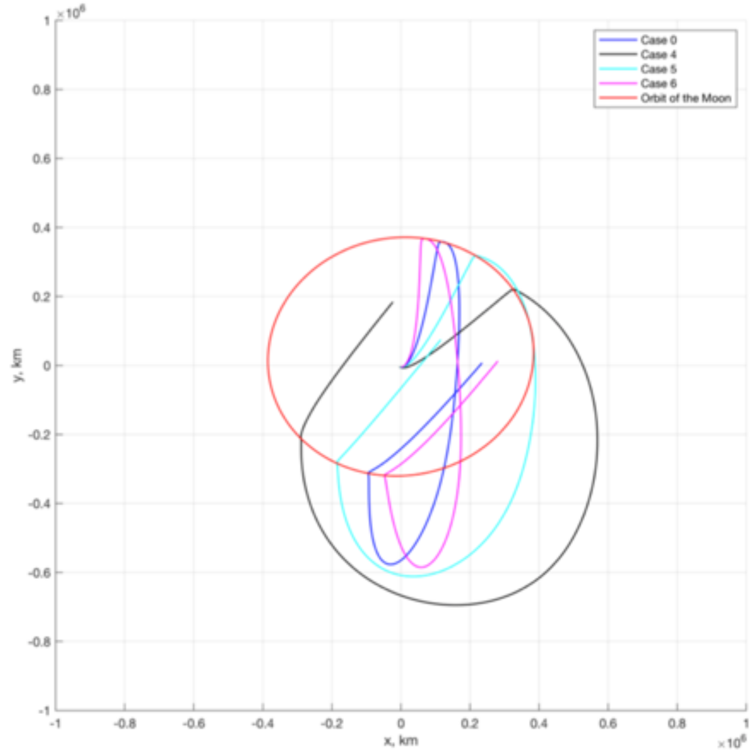

Figure 8. Comparison between solutions as the escape date varies, XY view

VII. Conclusions

In this section, we will find a summary of all the fundamental concepts that describe all the system, already analyzed in the previous sections.
To begin, we can underline the importance of the two flybys and the usefulness of each one. As well described, the two flybys happen after around 180 degrees because we have treated the backflip transfer. It is extremely important to notice that:

1) after the first flyby, the semi-major axis decreases. All this means a slight decrease in orbital energy. Although there is this energy loss, the first flyby it is necessary to adjust the inclination: for example, the inclination passes from 0.5 radians (about 28.5 degrees) to 1.78 radians (more than 100 degrees);
2) the second one instead, keeps about constant the inclination and increases significantly the energy.

Although the main goal is to increase the final energy, the first flyby, that instead reduces it, is necessary to achieve the right trajectory and the desired escape conditions.
The energy values, like all other values, dimensionless. For example, talking about the magnitude of energy of case 0, value has been increased from 0,00217 to 0,00758, with an increase of 250%. In another case, the increase reaches about 750%.
At the flybys, the S/C passes very close to the Moon. In particular the shortest distances are: for the first one, case 6, altitude of only 235 km while, for the second one, the closest is reached at case 4, with 334 km.
Another important aspect to take into account is the difference of the two model used in this document. The comparison between exact and approximated solutions shows that the results obtained through the analytical model are quite good solely for a preliminary analysis not too indepth. It provides you the whole trajectory with a fair accuracy and the final desired shape. It provides also dates, times, positions and velocities components. As already widely underlined, positions of S/C and Moon are not exact due to the hypothesis on which the model is based. However, a good approximation is ensured. In general, this solution is only used as tentative solution for more accurate models: these approximate solutions are necessary to achieve easily convergence in the more complex models.
Although this model has extremely simplified hypothesis, is important to notice that the relative positions between spacecraft and Moon, at both flybys, are kept. This is a key point to maintain the whole trajectory with the same shape.

Examining now the results, different considerations can be made. Talking about dates, as in Table 1, all the escape dates are in June 2022, while the departure takes place around April, same year. So the journey will last for about 2 months (on average 53 days). Most of this time is spent form one flyby to the other: this leg lasts for about 40 days, while from the second flyby to the escape it takes about a week. The longest leg, is characterized by an elliptical trajectory with a very high apogee (between $9 \cdot 10^5$ and $1 \cdot 10^6$ km).
Depending on the escape date, the trajectory in its entirety maintains almost constant the orbital parameters. Only change the angular positions at which the Moon is encountered because different times mean different Moon's positions along its orbit (increased escape date corresponds to a trajectory slightly rotated forward, counterclockwise). Analyzing Table 15 and Table 16, it can be noted that among the three cases, first flyby happens 2 days apart from each case. Remembering that Moon completes its orbit on average on 27 days, two days correspond to about 24 degrees. This angle is exactly the angular difference between Moon (and so S/C) position at the first flyby between one case to another.
That was the case of different escape dates. On the other hand, when the V_∞ is changed, it can be noted that an increase of 0.1 km/s connotes a reduction in the escape mass of only 18kg (0.2 %). Obviously, decreases also the time needed to reach the boundary of Earth's SOI.

Acknowledgments

The author wishes to express sincere gratitude to the thesis advisor Professor Lorenzo Casalino of the Mechanics and Aerospace department (DIMEAS) at Politecnico of Turin. He owes him a special thanks, because he was always available for every problem or doubt about this thesis and his help it was essential for the success of the work.
In addition, the author must express his very profound gratitude to his family for providing with unfailing support and continuous encouragement throughout his years of study and for the moving to Turin. This accomplishment would not have been possible without them.
Thank you so much.

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